

On the extended W -algebra of type \mathfrak{sl}_2 at positive rational level

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Abstract

The extended W -algebra of type \mathfrak{sl}_2 at positive rational level, denoted by \mathcal{M}_{p_+, p_-} , is a vertex operator algebra that was originally proposed in [FGSTb]. This vertex operator algebra is an extension of the minimal model vertex operator algebra and plays the role of symmetry algebra for certain logarithmic conformal field theories. We give a construction of \mathcal{M}_{p_+, p_-} in terms of screening operators and use this construction to prove that \mathcal{M}_{p_+, p_-} satisfies Zhu's c_2 -cofiniteness condition, calculate the structure of the zero mode algebra (also known as Zhu's algebra) and classify all simple \mathcal{M}_{p_+, p_-} -modules.

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1 Introduction

The theory of vertex operator algebras (VOA), which was developed by Borcherds [Bor], is an algebraic counterpart to conformal field theory and gives an algebraic meaning to the notions of locality and operator product expansions. For general facts about VOAs we refer to [FBZ, NTa, MNT].

Examples of conformal field theories on general Riemann surfaces for which vertex operator algebraic descriptions are known, are given by lattice VOAs, VOAs associated to integrable representations of affine Lie algebras and VOAs associated to minimal representations of the Virasoro algebra. The abelian categories associated to the representation theory of all these examples are semi-simple and the number of irreducible representations is finite.

In order to define conformal field theories over a Riemann surface associated to a VOA, the VOA needs to satisfy certain finiteness conditions. Zhu found such a finiteness condition [Zhu], which is now called Zhu's c_2 -cofiniteness condition. For a VOA satisfying Zhu's c_2 -cofiniteness condition it is known that the abelian category of its representations is both Noetherian and Artinian. Additionally the number of simple objects is finite. In general this abelian category is not semisimple [Zhu, FZ], though so far the semi-simple case is much better understood. Conformal field theories associated to VOAs with a non-semisimple representation theory are called logarithmic or non-semisimple.

Examples of VOAs which satisfy Zhu's c_2 -cofiniteness condition but not semisimplicity are given by so called W_p theories for which the representation category is by now well established [GK, GR, AMa, NTb, TW].

In this paper we analyse a different example which generalises the W_p theories. This example was originally defined in [FGSTb] and was called W_{p_+, p_-} . Unfortunately the letter "W" is rather overused in this context, so we will denote these VOAs by \mathcal{M}_{p_+, p_-} and call them "extended W -algebras of type \mathfrak{sl}_2 at rational level".

The \mathcal{M}_{p_+, p_-} are a family of VOAs parametrised by two coprime integers $p_+, p_- \geq 2$. They are defined by means of a lattice VOA \mathcal{V}_{p_+, p_-} and two screening operators S_+, S_- . The \mathcal{M}_{p_+, p_-} have the same central charge

$$c_{p_+, p_-} = 1 - 6 \frac{(p_+ - p_-)^2}{p_+ p_-},$$

as the minimal models. However, the Virasoro subtheory of \mathcal{M}_{p_+, p_-} is not isomorphic to the minimal model VOA MinVir_{p_+, p_-} ; rather, the minimal model VOA MinVir_{p_+, p_-} is obtained from \mathcal{M}_{p_+, p_-} by taking a quotient.

A number of results in this paper have already been described in [FGSTb]. In [FGSTb] the construction of integration cycles, over which products of screening operators are integrated, are described by using a Kazhdan-Lusztig type correspondence between the homology groups on configuration spaces of points the projective line with local coefficients and quantum groups at roots of unity. However, this correspondence is not yet well understood in this case [KL_a, KL_b, KL_c, KL_d, FGST_a]. The motivation for this paper was to reformulate the setup of [FGSTb] in a way intrinsic to VOAs without any reference to quantum groups. By developing VOA techniques, we have succeeded in not only proving the results of [FGSTb] but also going beyond that paper.

We introduce free field VOAs over the discrete valuation ring $\mathcal{O} = \mathbb{C}[[\varepsilon]]$ of formal complex power series. We discuss their representation theory, screening operators and the construction of cycles on which products of screening operators can be integrated. These integrated products of screening operators define intertwining operators of the Virasoro action on free field modules – called Fock modules – over \mathcal{O} , which allow us to explicitly compute all the data required for analysing the VOA structure of \mathcal{M}_{p_+, p_-} . Thus we are able to analyse the zero mode algebra of \mathcal{M}_{p_+, p_-} and prove that \mathcal{M}_{p_+, p_-} satisfies Zhu’s c_2 -cofiniteness condition. For $p_+ = 2$ and p_- odd, the c_2 -cofiniteness of \mathcal{M}_{p_+, p_-} has already been shown [AMb].

The results of this paper form a necessary starting point for studying problems such as the \mathcal{M}_{p_+, p_-} representation theory; the Kazhdan-Lusztig correspondence between \mathcal{M}_{p_+, p_-} and quantum groups; and conformal field theories with \mathcal{M}_{p_+, p_-} symmetry on the Riemann sphere and elliptic curves – as was done in [BPZ, TKb] for the semi-simple case – and more generally on moduli spaces of N point genus g stable curves.

This paper is organised as follows: In Section 2 we introduce some basic notation and definitions. We define VOAs with an emphasis on the Heisenberg and lattice VOAs as well as their screening operators. We also briefly explain how to construct the Poisson and zero mode algebra associated to a VOA as well as their implications for the representation theory of a VOA.

In Section 3 we construct cycles over which products of screening operators can be integrated. These integration cycles are elements of the homology groups of configuration spaces of N points on the projective line, with local coefficients defined by the monodromy of products of screening operators. Due to so called resonance problems homology and cohomology groups with these local coefficients exhibit very complicated behaviour [OT, Var]. To overcome these complications we deform the Heisenberg VOA, including its energy momentum tensor and screening operators, and construct the theory over the ring \mathcal{O} and its field of fractions $\mathcal{K} = \mathbb{C}((\varepsilon))$. The problem is thus

translated into constructing well behaved cycles such that all matrix elements of integrals of products of screening operators lie in \mathcal{O} rather than \mathcal{K} . We show this by using the theory of Jack polynomials [Mac]. By setting $\varepsilon = 0$ we then obtain integration cycles over \mathbb{C} for products of screening operators.

In Section 4 we construct all the singular vectors of Fock modules of the Virasoro algebra at central charge c_{p_+, p_-} and give a complete decomposition of these Fock modules as Virasoro modules. The theory of Fock modules of the Virasoro algebra at central charge c_{p_+, p_-} was developed by Feigin and Fuchs [FFa, FFb, FFc]. By means of the p_+ -th divided power operator $S_+^{[p_+]}$ and the p_- -th divided power operator $S_-^{[p_-]}$ we define intertwining operators E and F , called Frobenius maps. The maps E and F as well as their commutator $H = [E, F]$ are crucial to making calculations in \mathcal{M}_{p_+, p_-} tractable.

In Section 5 we show that the Frobenius maps E, F, H can be extended to be primary fields $E(z), F(z), H(z)$ of conformal dimension 1. We call these fields Frobenius currents. By using the properties of the Frobenius currents, we analyse the structure of \mathcal{M}_{p_+, p_-} as a module over itself. Then we can calculate the structure of the zero mode algebra of \mathcal{M}_{p_+, p_-} and show that \mathcal{M}_{p_+, p_-} satisfies Zhu's c_2 -cofiniteness condition. Finally we show that there exist a total of $\frac{1}{2}(p_+ - 1)(p_- - 1) + 2p_+p_-$ simple \mathcal{M}_{p_+, p_-} -modules. Of these simple modules $\frac{1}{2}(p_+ - 1)(p_- - 1)$ are isomorphic to the simple modules of the Virasoro minimal model VOA MinVir_{p_+, p_-} , the remaining $2p_+p_-$ simple modules are specific to \mathcal{M}_{p_+, p_-} .

In Section 6 we give our conclusions and state a list of future problems and conjectures associated to conformal field theories with \mathcal{M}_{p_+, p_-} symmetry.

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2 Basic definitions and notation

In this section we review basic definitions and notation for VOAs, in particular Heisenberg and lattice VOAs.

2.1 Vertex operator algebras

For a detailed discussion of vertex operator algebras see [FZ, NTa, MNT].

Definition 2.1. *A tuple $(V, |0\rangle, T, Y)$ is called a vertex operator algebra (VOA for short) where*

1. *V is a complex non-negative integer graded vector space*

$$V = \bigoplus_{n=0}^{\infty} V[n],$$

called the vacuum space of states.

2. *$|0\rangle \in V[0]$ is called the vacuum state.*
3. *$T \in V[2]$ is called the conformal vector.*
4. *Y is a \mathbb{C} -linear map*

$$Y : V \rightarrow \text{End}_{\mathbb{C}}(V)[[z, z^{-1}]]$$

called the vertex operator map.

These data are subject to the axioms:

1. *Each homogeneous subspace $V[n]$ of the space of states is finite dimensional and in particular $V[0]$ is spanned by the vacuum state.*
2. *For each $A \in V[h]$ there exists a Laurent expansion*

$$Y(A; z) = A(z) = \sum_{n \in \mathbb{Z}} A[n] z^{-n-h},$$

where $Y(A; z)$ is called a field and the $A[n]$ are called field modes. Then

$$Y(A; z)|0\rangle = A \in V[[z]]z$$

and

$$Y(|0\rangle; z) = \text{id}_V.$$

This is called the state field correspondence.

3. The field modes of the field corresponding to the conformal vector

$$Y(T; z) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

satisfy the commutation relations of the Virasoro algebra with fixed central charge $c = c_V$

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0}.$$

The field $T(z)$ is called the Virasoro field.

4. The zero mode of the Virasoro algebra L_0 acts semi-simply on V and the eigenvalues of L_0 define the grading of V , that is,

$$V[h] = \{A \in V \mid L_0 A = hA\}.$$

5. The Virasoro generator L_{-1} acts as the derivative with respect to z

$$\frac{d}{dz} Y(A; z) = Y(L_{-1} A; z),$$

for all $A \in V$.

6. For any two elements $A, B \in V$ the fields $Y(A; z)$ and $Y(B; w)$ are local, i.e. there exists a sufficiently large $N \in \mathbb{Z}$ such that

$$(z - w)^N [Y(A; z), Y(B; w)] = 0,$$

as elements of $\text{End}(V)[[z, z^{-1}, w, w^{-1}]]$.

7. For a homogeneous element $A \in V[h]$ and an element $B \in V$ the fields $Y(A; z)$ and $Y(B; w)$ satisfy the operator product expansion

$$\begin{aligned} Y(A; z)Y(B; w) &= Y(Y(A; z - w)B; w) \\ &= \sum_{n \in \mathbb{Z}} Y(A[n]B; w)(z - w)^{-n-h}. \end{aligned}$$

When there is no chance of confusion we will refer to a VOA just by its graded vector space V .

Remark 2.2. 1. For $A \in V[h]$ the Virasoro generators L_0 and L_{-1} satisfy

$$\begin{aligned} [L_{-1}, A[n]] &= -(n + h - 1)A[n - 1] \\ [L_0, A[n]] &= -nA[n]. \end{aligned}$$

2. The operator product expansion of the Virasoro field with itself is

$$T(z)T(w) = \frac{c_V/2}{(z-w)^4} + \frac{2}{(z-w)^2} + \frac{1}{z-w}\partial T(w) + \mathcal{O}(1).$$

3. For any homogeneous element $A \in V[h]$ such that $L_n A = 0$ for $n \geq 1$, the operator product expansion of the Virasoro field with $Y(A; z)$ is

$$T(z)Y(A; w) = \frac{h}{(z-w)^2}Y(A; w) + \frac{1}{z-w}\partial Y(A; w) + \mathcal{O}(1).$$

Such fields $Y(A; w)$ are called primary fields.

Definition 2.3. A VOA module M is a vector space that carries a representation Y_M

$$Y_M : V \rightarrow \text{End}_{\mathbb{C}}(M)[[z, z^{1-}]],$$

of the vertex operator Y , such that for all $A, B \in V$ and $C \in M$

1. $Y_M(A, z)C \in M((z))$,
2. $Y_M(|0\rangle; z) = \text{id}_M$, is the identity on M ,
3. $Y_M(A; z)Y_M(B; w) = Y_M(Y(A; z-w)B; w)$.

Let $O(V)$ be the vector subspace of V spanned by vectors

$$A \circ B = \text{Res}_{z=0} Y(A; z)B \frac{(1+z)^{h_A}}{z^2} dz = \sum_{n=0}^{h_A} \binom{h_A}{n} A[-n-1]B,$$

for $A \in V[h_A]$, $B \in V$. Furthermore, let $A * B$ be the binary operation

$$A * B = \text{Res}_{z=0} Y(A; z)B \frac{(1+z)^{h_A}}{z} dz = \sum_{n=0}^{h_A} \binom{h_A}{n} A[-n]B.$$

Proposition 2.4. The space $A_Z(V) = V/O(V)$ carries the structure of an associative \mathbb{C} algebra. Let $[A], [B] \in A_Z$ denote the classes represented by $A, B \in V$, then the multiplication in $A_Z(V)$ is given by

$$[A] \cdot [B] = [A * B].$$

1. The unit of A_Z is the coset of the vacuum state $1 = [|0\rangle]$.
2. The coset of the conformal vector $[T]$ lies in the centre of A_Z .

3. $(L_0 + L_{-1})A \in O(V)$ for all $A \in V$.
4. There is a 1 to 1 correspondence between simple A_Z -modules and simple V -modules. The simple A_Z -modules are isomorphic to the homogeneous space of lowest conformal weight of the corresponding simple V -module.
5. For $A \in V[h_A]$, $B \in V$ and $m \geq n \geq 0$

$$\text{Res}_{z=0} Y(A; z)B \frac{(1+z)^{h_A+n}}{z^{2+m}} dz \in O(V).$$

There is an equivalent definition of Zhu's algebra as a quotient of the algebra of zero modes. Let

$$U(V) = \bigoplus_{d \in \mathbb{Z}} U(V)[d]$$

$$U(V)[d] = \{P \in U(V) \mid [L_0, P] = -dP\}$$

be the graded associative algebra of modes of all the fields in V . Furthermore, let

$$F_p(V) = \bigoplus_{d \leq p} U(V)[d]$$

be a descending filtration of $U(V)$ and let

$$\mathcal{I} = \overline{F_{-1}(V) \cdot U(V)}$$

be the closure of the $U(V)$ -right ideal generated by $F_{-1}(V)$, then

$$I = \mathcal{I} \cap F_0(V)$$

is a closed two sided $F_0(V)$ ideal.

Definition 2.5. *The zero mode algebra is the quotient algebra*

$$A_0(V) = F_0(V)/I.$$

Proposition 2.6. 1. *There exists a canonical surjective \mathbb{C} algebra homomorphism*

$$U(V)[0] \rightarrow A_0(V)$$

which maps an element $P \in U(V)[0] \subset F_0(V)$ to its class in the zero mode algebra $A_0(V) = F_0(V)/I$.

2. There exists a well defined canonical isomorphism of \mathbb{C} algebras from Zhu's algebra $A_Z(V)$ to the zero mode algebra $A_0(V)$, such that for $A \in V$

$$\begin{aligned} A_Z(V) &\rightarrow A_0(V) \\ [A] &\mapsto [A[0]]. \end{aligned}$$

Remark 2.7. Since the zero mode algebra and Zhu's algebra are canonically isomorphic, we identify these two algebra and denote both by $A_0(V)$. For $A \in V$ we denote the corresponding class in $A_0(V)$ by $[A] = [A[0]]$.

Let $c_2(V)$ be the subspace of V given by

$$c_2(V) = \text{span}\{A[-(h_A + n)]B | n \geq 1\}.$$

Definition 2.8. The VOA V is said to satisfy Zhu's c_2 -cofiniteness condition if the quotient

$$V/c_2(V)$$

is finite dimensional.

Proposition 2.9. The quotient space $\mathfrak{p}(V) = V/c_2(V)$ carries the structure of a commutative Poisson algebra. Let $[A]_{\mathfrak{p}}, [B]_{\mathfrak{p}}$ be the cosets of $A \in V[h_A]$ and $B \in V$, then the multiplication and bracket are given by

$$\begin{aligned} [A]_{\mathfrak{p}} \cdot [B]_{\mathfrak{p}} &= [A[-h_A]B]_{\mathfrak{p}}, \\ \{[A]_{\mathfrak{p}}, [B]_{\mathfrak{p}}\} &= [A[-h_A + 1]B]_{\mathfrak{p}}. \end{aligned}$$

Remark 2.10. The space $O(V)$ is not spanned by homogeneous vectors therefore Zhu's algebra is not graded, it is merely filtered by conformal weight. The space $c_2(V)$, on the other hand, is spanned by homogeneous vectors, so the Poisson algebra $\mathfrak{p}(V)$ is graded by conformal weight.

Let $F_0(A_0(V)) \subset F_1(A_0(V)) \subset \dots$ be the filtration of $A_0(V)$ by conformal weight, that is

$$F_p(A_0(V)) = \left\{ [A] | A \in \bigoplus_{h=0}^p V[h] \right\}.$$

Then the gradification of $A_0(V)$ is the graded algebra

$$\text{Gr}(A_0(V)) = \sum_{p \geq 0} G_p(A_0(V)),$$

where

$$G_0(A_0(V)) = F_0(A_0(V)), \quad G_p(A_0(V)) = F_p(A_0(V))/F_{p-1}(A_0(V)), \quad p \geq 1.$$

Proposition 2.11. *There exists a surjection of graded \mathbb{C} algebras*

$$\mathfrak{p}(V) \rightarrow \text{Gr}(A_0(V)).$$

For proofs of the properties of Zhu's algebra see [Zhu, FZ] and for a proof of the existence of the canonical isomorphism between $A_Z(V)$ and $A_0(V)$ see [NTa].

2.2 The Heisenberg vertex operator algebra

The Heisenberg VOA is a central building block for all the VOAs considered in this paper. Before we can define the Heisenberg VOA, we must first define the Heisenberg algebra and its highest weight modules, called Fock modules.

Definition 2.12. 1. *Let $U(\mathfrak{b}_\pm)$ and $U(\mathfrak{b}_0)$ be \mathbb{Z} -graded polynomial algebras over \mathbb{C} given by*

$$\begin{aligned} U(\mathfrak{b}_\pm) &= \mathbb{C}[b_{\pm 1}, b_{\pm 2}, \dots] \\ U(\mathfrak{b}_0) &= \mathbb{C}[b_0]. \end{aligned}$$

The degree of b_n is $\deg(b_n) = -n$.

2. *The associative \mathbb{Z} -graded degree wise completed \mathbb{C} -algebra $U(\overline{\mathfrak{b}})$ is given by*

$$U(\overline{\mathfrak{b}}) = U(\mathfrak{b}_-) \hat{\otimes} U(\mathfrak{b}_+)$$

as a vector space, where $\hat{\otimes}$ denotes the degree wise completed tensor product. The algebra structure on $U(\overline{\mathfrak{b}})$ is defined by the Heisenberg commutation relations

$$[b_m, b_n] = m\delta_{m,-n} \cdot \text{id}, \quad m, n \in \mathbb{Z} \setminus \{0\}.$$

3. *The Heisenberg algebra is the \mathbb{Z} -graded associative algebra $U(\mathfrak{b})$ given by*

$$U(\mathfrak{b}) = U(\overline{\mathfrak{b}}) \otimes U(\mathfrak{b}_0)$$

and satisfies the commutation relations

$$[b_m, b_n] = m\delta_{m,-n} \cdot \text{id}, \quad m, n \in \mathbb{Z}.$$

Definition 2.13. *Let $\beta \in \mathbb{C}$.*

1. We define the left $U(\mathfrak{b})$ -module F_β – called a Fock module. It is generated by the state $|\beta\rangle$, which satisfies

$$\begin{aligned} b_n|\beta\rangle &= 0, \quad n \geq 1, \\ b_0|\beta\rangle &= \beta|\beta\rangle, \end{aligned}$$

such that

$$\begin{aligned} U(\mathfrak{b}_-) &\rightarrow F_\beta \\ P &\mapsto P|\beta\rangle \end{aligned}$$

is an isomorphism of complex vector spaces.

2. We define the right $U(\mathfrak{b})$ -module F_β^\vee – called a dual Fock module. It is generated by the state $\langle\beta|$, which satisfies

$$\begin{aligned} \langle\beta|b_n &= 0, \quad n \leq -1, \\ \langle\beta|b_0 &= \langle\beta|\beta, \end{aligned}$$

such that

$$\begin{aligned} U(\mathfrak{b}_+) &\rightarrow F_\beta^\vee \\ P &\mapsto \langle\beta|P \end{aligned}$$

is an isomorphism of complex vector spaces.

3. The two Fock modules F_β, F_β^\vee are equipped with an inner product

$$F_\beta^\vee \times F_\beta \rightarrow \mathbb{C},$$

defined by $\langle\beta|\beta\rangle = 1$.

The parameter β is called the Heisenberg weight.

Remark 2.14. Let \hat{b} be the conjugate of b_0 meaning that it satisfies the commutation relations

$$[b_m, \hat{b}] = \delta_{m,0}1.$$

For each $\gamma \in \mathbb{C}$, $e^{\gamma\hat{b}}$ defines a Heisenberg weight shifting map

$$e^{\gamma\hat{b}} : F_\beta \rightarrow F_{\beta+\gamma},$$

that satisfies

1. $e^{\gamma \hat{b}}|\beta\rangle = |\beta + \gamma\rangle$,
2. $e^{\gamma \hat{b}}$ commutes with $U(\mathfrak{b}_-)$ and $U(\mathfrak{b}_+)$.

For $\alpha_0 \in \mathbb{C}$ let

$$T = \frac{1}{2}(b_{-1}^2 - \alpha_0 b_{-2})|0\rangle \in F_0.$$

Proposition 2.15. *The Fock space F_0 carries the structure of a VOA, with*

$$\begin{aligned} Y(|0\rangle; z) &= \text{id}, \\ Y(b_{-1}|0\rangle; z) &= b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}, \\ Y(T; z) &= T(z) = \frac{1}{2}(: b(z)^2 : - \alpha_0 \partial b(z)). \end{aligned}$$

We denote this VOA by $\mathcal{F}_{\alpha_0} = (F_0, |0\rangle, T, Y)$.

Remark 2.16. 1. *In terms of Heisenberg generators the Virasoro generators are given by*

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : b_{n-k} b_k : - \frac{\alpha_0}{2} (n+1) b_n.$$

2. *The central charge of \mathcal{F}_{α_0} is given by*

$$c_{\alpha_0} = 1 - 3\alpha_0^2.$$

Proposition 2.17. 1. *The Fock module F_β is an irreducible \mathcal{F}_{α_0} -module for all $\beta \in \mathbb{C}$.*

2. *The abelian category of \mathcal{F}_{α_0} -modules, $\mathcal{F}_{\alpha_0}\text{-mod}$, is semisimple and the set of simple objects is given by $\{F_\beta\}_{\beta \in \mathbb{C}}$.*

3. *The generating state $|\beta\rangle$ of F_β satisfies*

$$\begin{aligned} L_n |\beta\rangle &= 0, \quad n \geq 1 \\ L_0 |\beta\rangle &= h_\beta |\beta\rangle, \end{aligned}$$

where

$$h_\beta = \frac{1}{2} \beta(\beta - \alpha_0).$$

We introduce an auxiliary field $\varphi(z)$, which is a formal primitive of $b(z)$

$$\varphi(z) = \hat{b} + b_0 \log z - \sum_{n \neq 0} \frac{b_n}{n} z^{-n}$$

and satisfies the operator product expansion

$$\varphi(z)\varphi(w) = \log(z-w) + \mathcal{O}(1).$$

Definition 2.18. For all $\beta \in \mathbb{C}$, let $V_\beta(z)$ denote the field

$$\begin{aligned} V_\beta(z) &=: e^{\beta\varphi(z)} := e^{\beta\hat{b}} z^{\beta b_0} \overline{V}_\beta(z) \\ \overline{V}_\beta(z) &= e^{\beta \sum_{n \geq 1} \frac{b_{-n}}{n} z^n} e^{-\beta \sum_{n \geq 1} \frac{b_n}{n} z^{-n}} \in \mathcal{U}(\overline{\mathfrak{b}}) \hat{\otimes} \mathbb{C}[z, z^{-1}], \end{aligned}$$

where $z^{\beta b_0} = \exp(b_0 \log(z))$.

Problems arising from the multivaluedness of $z^{\beta b_0}$ will be resolved in Section 4.

Proposition 2.19. For all $\beta \in \mathbb{C}$ the fields $V_\beta(z)$ satisfy:

1. For $\gamma \in \mathbb{C}$ the field $V_\beta(z)$ defines a map

$$V_\beta(z) : F_\gamma \rightarrow F_{\beta+\gamma}[[z, z^{-1}]] z^{\beta\gamma}.$$

2. The field $V_\beta(z)$ corresponds to the state $|\beta\rangle$, that is,

$$V_\beta(z)|0\rangle - |\beta\rangle \in F_\beta[[z]]z.$$

3. The field $V_\beta(z)$ is primary and satisfies the operator product expansion

$$T(z)V_\beta(w) = \frac{h_\beta}{(z-w)^2} V_\beta(w) + \frac{1}{z-w} \partial V_\beta(w) + \mathcal{O}(1).$$

4. For $\beta_1, \dots, \beta_k \in \mathbb{C}$ the k -fold product of fields $V_{\beta_i}(z_i)$ satisfies the operator product expansion

$$\prod_{i=1}^k V_{\beta_i}(w_i) = e^{\sum_{i=1}^k \beta_i \hat{b}} \prod_{i=1}^k w_i^{\beta_i b_0} \prod_{1 \leq i < j \leq k} (w_j - w_i)^{\beta_i \beta_j} : \prod_{i=1}^k \overline{V}_{\beta_i}(w_i) :,$$

where $:\prod_{i=1}^k \overline{V}_{\beta_i}(w_i):$ is an element of $\mathcal{U}(\overline{\mathfrak{b}}) \hat{\otimes} \mathbb{C}[z_1, z_1^{-1}, \dots, z_k, z_k^{-1}]$

$$: \prod_{i=1}^k \overline{V}_{\beta_i}(w_i) := e^{\sum_{i=1}^k \beta_i \sum_{n \geq 1} \frac{a_{-n}}{n} w_i^n} e^{-\sum_{i=1}^k \beta_i \sum_{n \geq 1} \frac{a_n}{n} w_i^{-n}}.$$

The Heisenberg algebra $\mathcal{U}(\mathfrak{h})$ admits an anti-involution

$$\begin{aligned}\sigma : \mathcal{U}(\mathfrak{h}) &\rightarrow \mathcal{U}(\mathfrak{h}) \\ b_n &\mapsto \delta_{n,0}\alpha_0 - b_{-n},\end{aligned}$$

such that

$$\sigma(L_n) = L_{-n}.$$

Remark 2.20. *The Virasoro generator L_0 is diagonalisable on the Fock spaces F_β . The eigenvalues of L_0 define a grading of F_β*

$$F_\beta = \bigoplus_{n \geq 0} F_\beta[h_\beta + n],$$

where

$$F_\beta[h] = \{u \in F_\beta \mid L_0 u = hu\}.$$

The dimension of these homogeneous subspaces is

$$\dim F_\beta[h_\beta + n] = p(n),$$

where $p(n)$ is the number of partitions of the integer n .

Definition 2.21. *The dual Fock module F_β^\vee is isomorphic to the graded dual space of F_β*

$$F_\beta^\vee = \bigoplus_{n \geq 0} \text{Hom}(F_\beta[h_\beta + n], \mathbb{C}).$$

The anti-involution σ induces the structure of a left $U(\mathfrak{h})$ -module on F_β^\vee by

$$\langle b_n \varphi, u \rangle = \langle \varphi, \sigma(a_n)u \rangle$$

for all $n \in \mathbb{Z}$, $\varphi \in F_\beta^*$, $u \in F_\beta$. We denote this left module by F_β^* and call it the contragredient dual of F_β .

Proposition 2.22. *Taking the contragredient defines a contravariant functor*

$$* : \mathcal{F}_{\alpha_0}\text{-mod} \rightarrow \mathcal{F}_{\alpha_0}\text{-mod},$$

satisfying

$$F_\beta^* = F_{\alpha_0 - \beta},$$

such that $(F_\beta^*)^* = F_\beta$.

2.3 The lattice vertex operator algebra \mathcal{V}_{p_+, p_-}

The lattice VOA \mathcal{V}_{p_+, p_-} is defined for special values of the parameter α_0 and by restricting the weights of the Fock spaces to a certain lattice. Let $p_+, p_- \geq 2$ be two coprime integers, such that

$$\begin{aligned} \alpha_+ &= \sqrt{\frac{2p_-}{p_+}} & \alpha_- &= -\sqrt{\frac{2p_+}{p_-}} \\ \kappa_+ &= \frac{\alpha_+^2}{2} = \frac{p_-}{p_+} & \kappa_- &= \frac{\alpha_-^2}{2} = \frac{p_+}{p_-} \\ \alpha_0 &= \alpha_+ + \alpha_- & \alpha &= p_+ \alpha_+ = -p_- \alpha_- . \end{aligned}$$

The parameters $\kappa_+ = \kappa_-^{-1}$ are called the level of \mathcal{V}_{p_+, p_-} . This is the positive rational level mentioned in the title of this paper. Next we define the rank 1 lattices

$$Y = \mathbb{Z}\sqrt{2p_+p_-} \quad X = \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}) = \mathbb{Z}\frac{1}{\sqrt{2p_+p_-}} .$$

Both α_+ and α_- lie in X and give rise to the parametrisation

$$\beta_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_- .$$

Note that $\beta_{r,s} = \beta_{r+p_+, s+p_-}$ and we use the shorthand

$$\beta_{r,s;n} = \beta_{r-np_+, s} = \beta_{r,s+np_-} .$$

When denoting the weights of Fock spaces, we will only write the indices and drop the “ β ” from $\beta_{r,s;n}$ or $\beta_{r,s}$, that is

$$F_{\beta_{r,s;n}} = F_{r,s;n} , \quad F_{\beta_{r,s}} = F_{r,s} .$$

Remark 2.23. 1. For $\alpha_0 = \alpha_+ + \alpha_-$ we denote the Heisenberg VOA by \mathcal{F}_{p_+, p_-} instead of \mathcal{F}_{α_0} .

2. Taking the contragredient of a Fock space $F_{r,s;n}$ reverses the sign of the indices:

$$F_{r,s;n}^* = F_{-r, -s; -n} .$$

Definition 2.24. The lattice VOA \mathcal{V}_{p_+, p_-} is the tuple $(V_{[0]}, |0\rangle, \frac{1}{2}(a_{-1}^2 + \alpha_0 b_2)|0\rangle, Y)$, where

1. the underlying vector space of \mathcal{V}_{p_+, p_-} is given by

$$V_{[0]} = \bigoplus_{\beta \in Y} F_\beta = \bigoplus_{n \in \mathbb{Z}} F_{n\alpha},$$

2. the vacuum and conformal vectors are those of \mathcal{F}_{p_+, p_-} ,

3. the vertex operator map is characterised by

$$\begin{aligned} Y(|0\rangle; z) &= \text{id} & Y(|\beta\rangle; z) &= V_\beta(z), \quad \beta \in Y \\ Y(b_{-1}|0\rangle) &= b(z) & Y(T; z) &= \frac{1}{2} : b(z)^2 : + \frac{\alpha_0}{2} \partial b(z), \end{aligned}$$

Remark 2.25. The relations for the vertex operator map in the definition above uniquely define the VOA structure of \mathcal{V}_{p_+, p_-} .

1. The central charge of the Virasoro field $T(z)$ is

$$c_{p_+, p_-} = 1 - 6 \frac{(p_+ - p_-)^2}{p_+ p_-}.$$

2. The zero mode of the Virasoro field $T(z)$ is given by

$$L_0 = \frac{1}{2} b_0^2 - \frac{1}{2} \alpha_0 b_0 + \sum_{n \geq 1} b_{-n} b_n.$$

Therefore the conformal weight of the generating state $|\beta\rangle$ of a Fock module F_β is

$$h_\beta = \frac{1}{2} \beta(\beta - \alpha_0).$$

3. We define $h_{r,s} = h_{\beta_{r,s}}$, $r, s \in \mathbb{Z}$, then we have

$$h_{r,s} = \frac{r^2 - 1}{4} \kappa_+ - \frac{rs - 1}{2} + \frac{s^2 - 1}{4} \kappa_-.$$

Proposition 2.26. The abelian category $\mathcal{V}_{p_+, p_-}\text{-mod}$ of \mathcal{V}_{p_+, p_-} -modules is semi-simple with $2p_+ p_-$ simple objects. These simple objects are parametrised by the cosets of X/Y

$$V_{[\beta]} = \bigoplus_{\gamma \in \beta + Y} F_\gamma, \quad \beta \in X.$$

Remark 2.27. Using the $\beta_{r,s;n}$ we parametrise the simple \mathcal{V}_{p_+,p_-} -modules as

$$V_{r,s}^+ = \bigoplus_{n \in \mathbb{Z}} F_{r,s;2n}$$

$$V_{r,s}^- = \bigoplus_{n \in \mathbb{Z}} F_{r,s;2n+1} ,$$

for $1 \leq r \leq p_+$, $1 \leq s \leq p_-$. In this notation $V_{[0]} = V_{1,1}^+$.

By the formula for conformal weights in Proposition 2.17, the two Heisenberg weights α_+, α_- have conformal weight $h_{\alpha_{\pm}} = 1$. These are the only Heisenberg weights with conformal weight 1. We call the fields corresponding to $|\alpha_{\pm}\rangle$ screening operators and denote them by

$$S_{\pm}(z) =: e^{\alpha_{\pm}\varphi(z)} : .$$

Since $h_{\alpha_{\pm}} = 1$, these fields define intertwining operators, that is, the map

$$S_{\pm} = \oint S_{\pm}(z) dz : V_{[0]} \rightarrow V_{[\alpha_{\pm}]}$$

is \mathbb{C} -linear and commutes with the Virasoro algebra. Note that since $\alpha_{\pm} \notin Y$ the fields $S_{\pm}(z)$ do not belong to \mathcal{V}_{p_+,p_-} . We will later define the extended W -algebra \mathcal{M}_{p_+,p_-} as the subVOA of \mathcal{V}_{p_+,p_-} given by the intersection of the kernels of S_+ and S_- .

3 Deformation of screening operators

Over the course of this paper it will be necessary to consider integrals of products of screening operators and not just the residues of individual screening operators. In order to perform these integrals one needs to consider homology groups of configuration spaces of N points on the projective line with local coefficients. It is necessary to use local coefficients because these products of screening operators are not single valued, but have non-trivial monodromies that are roots of unity. The homology groups with such local coefficients exhibit very complicated behaviour and in order to make them tractable we deform the Heisenberg VOA, that is, we deform its conformal structure and its screening operators. The associated local system then no longer exhibits monodromy at roots of unity and the homology groups of these deformed local system are very simple. After analysing the deformed case in detail, we will show that one can take a meaningful limit to the undeformed case.

3.1 Deformation of the Heisenberg vertex operator algebra

Let $\mathcal{O} = \mathbb{C}[[\varepsilon]]$ be the ring of formal power series with coefficients in \mathbb{C} and let $\mathcal{K} = \mathbb{C}((\varepsilon))$ be the fraction field of \mathcal{O} . We enlarge the ground field \mathbb{C} of the Heisenberg algebra $U(\mathfrak{b})$ introduced in Section 2.2 to the rings \mathcal{O} and \mathcal{K} .

Definition 3.1. Let ${}_{\mathcal{K}}U(\mathfrak{b}_{\pm})$ and ${}_{\mathcal{O}}U(\mathfrak{b}_{\pm})$ be the Heisenberg algebra over \mathcal{K} and \mathcal{O} respectively, that is

$$\begin{aligned} {}_{\mathcal{K}}U(\mathfrak{b}_{\pm}) &= \mathcal{K}[b_{\pm 1}, b_{\pm 2}, \dots] \\ {}_{\mathcal{K}}U(\mathfrak{b}_0) &= \mathcal{K}[b_0] \\ {}_{\mathcal{K}}U(\bar{\mathfrak{b}}) &= {}_{\mathcal{K}}U(\mathfrak{b}_-) \hat{\otimes}_{\mathcal{K}} {}_{\mathcal{K}}U(\mathfrak{b}_+) \\ &= \bigoplus_{d \in \mathbb{Z}} {}_{\mathcal{K}}U(\bar{\mathfrak{b}})[d] \\ {}_{\mathcal{K}}U(\mathfrak{b}) &= {}_{\mathcal{K}}U(\bar{\mathfrak{b}}) \otimes_{\mathcal{K}} {}_{\mathcal{K}}U(\mathfrak{b}_0) \end{aligned}$$

and

$$\begin{aligned} {}_{\mathcal{O}}U(\mathfrak{b}_{\pm}) &= \mathcal{O}[b_{\pm 1}, b_{\pm 2}, \dots] \\ {}_{\mathcal{O}}U(\mathfrak{b}_0) &= \mathcal{O}[b_0] \\ {}_{\mathcal{O}}U(\bar{\mathfrak{b}}) &= {}_{\mathcal{O}}U(\mathfrak{b}_-) \hat{\otimes}_{\mathcal{O}} {}_{\mathcal{O}}U(\mathfrak{b}_+) \\ &= \bigoplus_{d \in \mathbb{Z}} {}_{\mathcal{O}}U(\bar{\mathfrak{b}})[d] \\ {}_{\mathcal{O}}U(\mathfrak{b}) &= {}_{\mathcal{O}}U(\bar{\mathfrak{b}}) \otimes_{\mathcal{O}} {}_{\mathcal{O}}U(\mathfrak{b}_0). \end{aligned}$$

The Heisenberg algebra over \mathcal{K} contains the \mathcal{O} subalgebra ${}_{\mathcal{O}}U(\mathfrak{b})$ as an \mathcal{O} lattice.

We deform the parameters α_{\pm} and κ_{\pm} as follows. Let

$$\alpha_{\pm}(\varepsilon) = \alpha_{\pm}^{(0)} + \alpha_{\pm}^{(1)}\varepsilon + \alpha_{\pm}^{(2)}\varepsilon^2 + \dots \in \mathcal{O}$$

with $\alpha_{\pm}^{(0)} = \alpha_{\pm}$, $\alpha_{\pm}^{(1)} \neq 0$ and $\alpha_+(\varepsilon)\alpha_-(\varepsilon) = -2$. Furthermore, let

$$\begin{aligned} \kappa_{\pm}(\varepsilon) &= \frac{1}{2}\alpha_{\pm}(\varepsilon)^2 \in \mathcal{O}, \\ \alpha_0(\varepsilon) &= \alpha_+(\varepsilon) + \alpha_-(\varepsilon) \in \mathcal{O}. \end{aligned}$$

We define the rank 2 abelian group

$${}_{\mathcal{O}}X = \mathbb{Z}\frac{\alpha_+(\varepsilon)}{2} \oplus \mathbb{Z}\frac{\alpha_-(\varepsilon)}{2} \subset \mathcal{O}$$

and for $(r, s) \in \mathbb{Z}^2$

$$\beta_{r,s}(\varepsilon) = \frac{1-r}{2}\alpha_+(\varepsilon) + \frac{1-s}{2}\alpha_-(\varepsilon) \in {}_{\mathcal{O}}X.$$

Definition 3.2. 1. For each $\beta \in {}_{\mathcal{O}}X$ we define the left ${}_{\mathcal{K}}U(b)$ -module ${}_{\mathcal{K}}F_{\beta}$ generated by $|\beta\rangle$

$$\begin{aligned} b_n|\beta\rangle &= 0, \quad n \geq 1 \\ b_0|\beta\rangle &= \beta|\beta\rangle, \end{aligned}$$

such that

$$\begin{aligned} {}_{\mathcal{K}}U(b_-) &\rightarrow {}_{\mathcal{K}}F_{\beta} \\ P &\mapsto P|\beta\rangle \end{aligned}$$

is an isomorphism of \mathcal{K} -vector spaces.

2. Let ${}_{\mathcal{O}}F_{\beta}$ be the subspace of ${}_{\mathcal{K}}F_{\beta}$ given by

$${}_{\mathcal{O}}F_{\beta} = {}_{\mathcal{O}}U(\mathfrak{b})|\beta\rangle.$$

The energy momentum tensor and other fields are defined in the same way as in the previous section

$$\begin{aligned} T(z) &= \frac{1}{2} : b(z)^2 : + \frac{\alpha_0(\varepsilon)}{2} \partial b(z) \\ V_{\beta}(z) &=: e^{\beta\varphi(z)} :, \end{aligned}$$

with β now in \mathcal{O} instead of \mathbb{C} . By evaluating operator product expansions it follows that the central charge and conformal weights are given by the same formulae as before

$$\begin{aligned} c_{p_+, p_-}(\varepsilon) &= 1 - 3\alpha_0(\varepsilon)^2 \in \mathcal{O}, \\ h_{\beta}(\varepsilon) &= \frac{1}{2}\beta(\beta - \alpha_0(\varepsilon)) \in \mathcal{O}. \end{aligned}$$

Set $h_{r,s}(\varepsilon) = h_{\beta_{r,s}(\varepsilon)}$, $r, s \in \mathbb{Z}$, then

$$h_{r,s}(\varepsilon) = \frac{r^2 - 1}{4}\kappa_+(\varepsilon) - \frac{rs - 1}{2} + \frac{s^2 - 1}{4}\kappa_-(\varepsilon).$$

Proposition 3.3. Let ${}_{\mathcal{K}}\mathcal{F}(p_+, p_-) = ({}_{\mathcal{K}}F_0, |0\rangle, \frac{1}{2}(b_{-1}^2 - \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$, then ${}_{\mathcal{K}}\mathcal{F}(p_+, p_-)$ has the structure of a VOA over the field \mathcal{K} .

Proposition 3.4. For each $A \in {}_{\mathcal{O}}F_0$, the field $Y(A; z)$ preserves the \mathcal{O} lattice ${}_{\mathcal{O}}F_0$ of ${}_{\mathcal{K}}F_0$, that is

$$Y(A; z) \in \text{End}_{\mathcal{O}}({}_{\mathcal{O}}F_0)[[z, z^{-1}]].$$

For each $(r, s) \in \mathbb{Z}^2$ we use the shorthand ${}_{\mathcal{K}}F_{\beta_{r,s}} = {}_{\mathcal{K}}F_{r,s}$ and ${}_{\mathcal{O}}F_{\beta_{r,s}} = {}_{\mathcal{O}}F_{r,s}$.

The two Heisenberg weights $\alpha_{\pm}(\varepsilon)$ have conformal weight $h_{\alpha_{\pm}}(\varepsilon) = 1$. Therefore fields

$$\begin{aligned} S_+(z) &=: e^{\alpha_+(\varepsilon)\varphi(z)} : \\ S_-(z) &=: e^{\alpha_-(\varepsilon)\varphi(z)} : \end{aligned}$$

define screening operators for ${}_{\mathcal{K}}\mathcal{F}(p_+, p_-)$.

3.2 The construction of renormalisable cycles

In this section we construct cycles over which we can integrate products of the screening operators $S_+(z)$ and $S_-(z)$.

Consider the N -fold product of $S_{\pm}(z)$

$$\prod_{i=1}^N S_{\pm}(z_i) = e^{N\alpha_{\pm}(\varepsilon)\hat{b}} \prod_{i=1}^N z_i^{\alpha_{\pm}(\varepsilon)b_0} \prod_{1 \leq i \neq j \leq N} (z_i - z_j)^{\alpha_{\pm}(\varepsilon)^2/2} : \prod_{i=1}^N \bar{S}_{\pm}(z_i) :,$$

where $: \prod_{i=1}^N \bar{S}_{\pm}(z_i) : \in {}_{\mathcal{O}}U(\bar{\mathfrak{b}}) \hat{\otimes} {}_{\mathcal{O}}\mathcal{O}[[z_1^{\pm}, \dots, z_n^{\pm}]]$

$$: \prod_{i=1}^N \bar{S}_{\pm}(z_i) := \prod_{k \geq 0} e^{\alpha_{\pm}(\varepsilon) \sum_{i=1}^N \frac{z_i^k}{k} a_{-k}} \prod_{k \geq 0} e^{-\alpha_{\pm}(\varepsilon) \sum_{i=1}^N \frac{z_i^{-k}}{k} a_k}.$$

Note that $: \prod_{i=1}^N \bar{S}_{\pm}(z_i) :$ is symmetric with respect to permuting the variables z_i .

Let $r \geq 1, s \in \mathbb{Z}$, then if we evaluate the operator $S_+(z_1) \cdots S_+(z_r)$ on ${}_{\mathcal{K}}F_{r,s}$ we have

$$\prod_i^r S_+(z_i) = e^{r\alpha_+(\varepsilon)\hat{b}} \Delta_r(z_1, \dots, z_r; \kappa_-(\varepsilon)) \prod_{i=1}^r z_i^{s-1} : \prod_i^r \bar{S}_+(z_i) :.$$

where

$$\Delta_r(z_1, \dots, z_r; \kappa_-(\varepsilon)) = \prod_{1 \leq i \neq j \leq r} (1 - \frac{z_i}{z_j})^{1/\kappa_-(\varepsilon)}.$$

Analogously let $s \geq 1, r \in \mathbb{Z}$, if we then evaluate $S_-(z_1) \cdots S_-(z_s)$ on ${}_{\mathcal{K}}F_{r,s}$ we have

$$\prod_i^s S_-(z_i) = e^{s\alpha_-(\varepsilon)\hat{b}} \Delta_s(z_1, \dots, z_s; \kappa_+(\varepsilon)) \prod_{i=1}^s z_i^{r-1} : \prod_i^s \bar{S}_-(z_i) :$$

where

$$\Delta_s(z_1, \dots, z_s; \kappa_+(\varepsilon)) = \prod_{1 \leq i \neq j \leq s} \left(1 - \frac{z_i}{z_j}\right)^{1/\kappa_+(\varepsilon)}.$$

For the remainder of this subsection the goal will be to find cycles over which we can integrate the multivalued functions $\Delta_N(z_1, \dots, z_N; \kappa)$.

Consider the N dimensional complex manifold

$$X_N = \{(z_1, \dots, z_N) \in (\mathbb{C}^*)^N; z_i \neq z_j \text{ for } i \neq j\}.$$

and fix

$$\kappa = \kappa_0 + \kappa_1 \varepsilon + \dots \in \mathcal{O}$$

with

$$\kappa_0 \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}, \quad \kappa_1 \neq 0.$$

Then

$$\Delta_N(z; \kappa) = \prod_{1 \leq i \neq j \leq N} \left(1 - \frac{z_i}{z_j}\right)^{1/\kappa}$$

defines a multivalued holomorphic function on X_N . We define the single valued 1-form

$$\omega_N(\kappa) = d \log \Delta_N(z; \kappa) = \frac{1}{\kappa} \sum_{1 \leq i \neq j \leq N} \frac{d(z_i - z_j)}{z_i - z_j} + (1 - N) \frac{1}{\kappa} \sum_{i=1}^N \frac{d z_i}{z_i}.$$

Then we can define the twisted de Rham complex $(\mathcal{K}\Omega_{X_N}^p, \nabla_{\omega_N(\kappa)})$ by

$$\nabla_{\omega_N(\kappa)} = d + \omega_N(\kappa) \wedge.$$

For an introduction to twisted de Rham theory we refer the reader to [AK]. The differential $\nabla_{\omega_N(\kappa)}$ defines the twisted de Rham cohomology groups

$$\mathcal{K}H^p(X_N, \nabla_{\omega_N(\kappa)})$$

as well as their \mathcal{O} -lattices

$$\mathcal{O}H^p(X_N, \nabla_{\omega_N(\kappa)}).$$

Let ${}_{\mathcal{O}}\mathcal{L}_N(\kappa)$ be the local system over the ring \mathcal{O} determined by the monodromy of $\Delta_N(z; \kappa)$ and let ${}_{\mathcal{O}}\mathcal{L}_N(\kappa)^\vee = \text{Hom}_{\mathcal{O}}({}_{\mathcal{O}}\mathcal{L}_N, \mathcal{O})$ be the dual local system. We also define

$$\begin{aligned}\kappa\mathcal{L}_N(\kappa) &= {}_{\mathcal{O}}\mathcal{L}_N(\kappa) \otimes_{\mathcal{O}} \mathcal{K} \\ \kappa\mathcal{L}_N(\kappa)^\vee &= {}_{\mathcal{O}}\mathcal{L}_N(\kappa)^\vee \otimes_{\mathcal{O}} \mathcal{K}.\end{aligned}$$

By the theorem of twisted de Rham theory [AK, Chapter 2]

$$\kappa H^p(X_N, \nabla_{\omega_N(\kappa)}) \cong \kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa))$$

and

$$\kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa)) \cong \text{Hom}_{\mathcal{K}}(\kappa H_p(X_N, \kappa\mathcal{L}_N^\vee(\kappa)), \mathcal{K})$$

as \mathcal{K} -vector spaces.

Since the symmetric group acts in a compatible fashion on both X_N and ${}_{\mathcal{O}}\mathcal{L}_N(\kappa)$, the cohomology group $\kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa))$ carries the structure of a finite dimensional representation of S_N . We can therefore decompose $\kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa))$ into a direct sum of irreducible S_N modules. For any S_N module M , let $M^{S_N^-}$ be the skew symmetric part of M . It is known that [AK, Chapter 2]

$$\begin{aligned}\kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa)) &= 0 \quad \text{for } p > N, \\ {}_{\mathcal{O}}H^p(X_N, {}_{\mathcal{O}}\mathcal{L}_N(\kappa)) &= 0 \quad \text{for } p > N.\end{aligned}$$

We are interested in the skew symmetric parts of the N th cohomology groups, for which it is known that [Var]

$$\begin{aligned}\kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa))^{S_N^-} &\cong \mathcal{K} \\ {}_{\mathcal{O}}H^p(X_N, {}_{\mathcal{O}}\mathcal{L}_N(\kappa))^{S_N^-} &\cong \mathcal{O},\end{aligned}$$

and

$${}_{\mathcal{O}}H^p(X_N, {}_{\mathcal{O}}\mathcal{L}_N(\kappa))^{S_N^-} \otimes_{\mathcal{O}} \mathcal{K} = \kappa H^p(X_N, \kappa\mathcal{L}_N(\kappa))^{S_N^-}.$$

Next we sketch the construction of a cycle $[\Gamma_N(\kappa)] \in \kappa H_N(X_N, \kappa\mathcal{L}_N(\kappa))$ that has a non-trivial component in $\kappa H_N(X_N, \kappa\mathcal{L}_N(\kappa))^{S_N^-}$. This construction was first done in [TKa]. We merely summarise the argument.

Proposition 3.5. *There exists a construction of a cycle*

$$[\Delta_N(\kappa)] \in \kappa H_N(X_N, \kappa\mathcal{L}_N(\kappa))$$

such that

$$\int_{[\Delta_N(\kappa)]} \Delta_N(z; \kappa) \prod_{i=1}^N \frac{dz_i}{z_i} = c_N(\kappa),$$

where $c_N(\kappa)$ is a constant given by Selberg integrals

$$c_N(\kappa) = \frac{1}{(k-1)!} \prod_{i=1}^{k-1} \frac{\Gamma((i-k)\kappa)\Gamma((i+1)\kappa+1)}{\Gamma(\kappa+1)}.$$

The construction of the cycle $[\Delta_N(\kappa)]$ is as follows. At first we set

$$Y_{N-1} = \{(y_1, \dots, y_{N-1}) \in (\mathbb{C}^*)^{N-1} | y_i \neq y_j, y_i \neq 1\}$$

Then there exists a holomorphic isomorphism

$$\begin{aligned} \mathbb{C}^* \times Y_{N-1} &\rightarrow X_N \\ (z, y_1, \dots, y_{N-1}) &\mapsto (z, zy_1, \dots, zy_{N-1}). \end{aligned}$$

This homomorphism commutes with $S_{N-1} \subset S_N$. When written with respect to (z, y_1, \dots, y_N) the multivalued function $\Delta_N(z, \kappa)$ is given by

$$\overline{\Delta_N}(y; \kappa) = \prod_{i=1}^{N-1} (1 - y_i)^{2/\kappa} \prod_{1 \leq i \neq j \leq N-1} (y_i - y_j)^{1/\kappa} \prod_{i=1}^{N-1} y_i^{(1-N)/\kappa},$$

which is independent of z . So the homology groups on X_N factorise as

$${}_{\mathcal{K}}H_N(X_N, {}_{\mathcal{K}}\mathcal{L}_N^{\vee}(\kappa)) \cong H_1(\mathbb{C}^*, \mathbb{C}) \otimes {}_{\mathcal{K}}H_{N-1}(Y_{N-1}, {}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa)),$$

where ${}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa)$ is the dual of the local system on Y_{N-1} determined by $\overline{\Delta_N}(y; \kappa)$. It is known that [AK, Chapter 2]

$${}_{\mathcal{K}}H_p(Y_{N-1}, {}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa)) \rightarrow H_p^{l.f.}(Y_{N-1}, {}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa))$$

is an isomorphism of homology groups, where $H_p^{l.f.}(Y_{N-1}, {}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa))$ is the homology group of locally finite cycles with coefficients in ${}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa)$. Then we construct cycles

$$\begin{aligned} [\overline{\Delta_{N-1}}; \varphi] &\in H_{N-1}^{l.f.}(Y_{N-1}, {}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa)) \\ \overline{\Delta_{N-1}} &= \{1 > y_1 > \dots > y_{N-1} > 0\} \subset Y_{N-1}, \end{aligned}$$

where φ is the principal branch of $\overline{\Delta}_N(y; \kappa)$ in $\overline{\Delta}_{N-1}$. Therefore there exist cycles $[\overline{\Delta}_{N-1}(\kappa)] \in {}_{\mathcal{K}}H_{N-1}(Y_{N-1}, {}_{\mathcal{K}}\overline{\mathcal{L}}_{N-1}^{\vee}(\kappa))$ which correspond to $[\overline{\Delta}_{N-1}]$ by the above isomorphism. Such cycles were constructed in [TKa] by using a blow up $\hat{Y}_{N-1} \rightarrow Y_{N-1}$. Then $[\Delta_N(\kappa)] \in {}_{\mathcal{K}}H_N(X_N, {}_{\mathcal{K}}\mathcal{L}_N^{\vee}(\kappa))$ is given by

$$[\Delta_N(\kappa)] = \text{Res}_{z=0} \, dz \otimes [\overline{\Delta}_{N-1}(\kappa)],$$

that is, taking the residue in z and integrating along $[\overline{\Delta}_{N-1}(\kappa)]$ where

$$\int_{[\overline{\Delta}_{N-1}(\kappa)]} \overline{\Delta}_N(y; \kappa) \prod_{i=1}^{N-1} \frac{dy_i}{y_i} = c_N(\kappa) \quad (3.1)$$

is precisely the Selberg integral.

Let $[\Gamma(\kappa)]$ be the renormalised cycle

$$[\Gamma_N(\kappa)] = \frac{1}{c_N(\kappa)} [\Delta_N(\kappa)],$$

such that

$$\int_{[\Gamma_N(\kappa)]} \Delta_N(z; \kappa) \prod_{i=1}^N \frac{dz_i}{z_i} = 1.$$

Definition 3.6. For $f \in \mathcal{K}[z_1^{\pm}, \dots, z_N^{\pm}]^{S_N}$ the cycle $[\Gamma_N(\kappa)]$ defines a \mathcal{K} -linear map

$$\langle \rangle_{\kappa}^N : \mathcal{K}[z_1^{\pm}, \dots, z_N^{\pm}]^{S_N} \rightarrow \mathcal{K}$$

by the formula

$$\langle f(z) \rangle_{\kappa}^N = \int_{[\Delta_N(\kappa)]} \Delta_N(z; \kappa) f(z) \prod_{i=1}^N \frac{dz_i}{z_i}.$$

Proposition 3.7. 1. $\langle 1 \rangle_{\kappa}^N = 1$

2. $\langle f \rangle_{\kappa}^N = 0$ if $\deg f \neq 0$

3. $\langle f \rangle_{\kappa}^N = \langle \overline{f} \rangle_{\kappa}^N$ where $\overline{f}(z_1, \dots, z_N) = f(z_1^{-1}, \dots, z_N^{-1})$.

Definition 3.8. Let $(\ , \)_{\kappa}^N$ be the bilinear \mathcal{K} -form

$$(\ , \)_{\kappa}^N : \mathcal{K}[z_1, \dots, z_N]^{S_N} \otimes_{\mathcal{K}} \mathcal{K}[z_1, \dots, z_N]^{S_N} \rightarrow \mathcal{K}$$

defined by

$$(f, g)_{\kappa}^N = \langle \overline{f}g \rangle_{\kappa}^N.$$

Following Macdonald's book [Mac], we will evaluate this bilinear form in the next section by using the theory of Jack polynomials. We will also be able to show that it defines an inner product of symmetric polynomials over \mathcal{O} , that is

$$(\cdot, \cdot)_{\kappa}^N : \mathcal{O}[z_1, \dots, z_N]^{S_N} \otimes_{\mathcal{O}} \mathcal{O}[z_1, \dots, z_N]^{S_N} \rightarrow \mathcal{O}.$$

3.3 The theory of Jack polynomials

We introduce the theory of Jack polynomials following Macdonald's book [Mac, Chapter 6]. Fix the parameter κ to be

$$\kappa = \kappa_0 + \kappa_1 \varepsilon + \dots \in \mathcal{O},$$

such that $\kappa_0 \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ and $\kappa_1 \neq 0$.

Definition 3.9. The rings of symmetric polynomials over \mathcal{O} and \mathcal{K} in N variables x_i are given by

$$\begin{aligned} \kappa \Lambda_N &= \mathcal{K}[x_1, \dots, x_N]^{S_N} & \mathcal{O} \Lambda_N &= \mathcal{O}[x_1, \dots, x_N]^{S_N} \\ &= \mathcal{K}[p_1, \dots, p_N] & &= \mathcal{O}[p_1, \dots, p_N] \end{aligned}$$

where

$$p_n = \sum_{i \geq 1} x_i^n,$$

are power sums. The symmetric polynomials $\kappa \Lambda_N$ form a graded commutative algebra with $\deg p_n = n$. Rings of symmetric polynomials in different numbers of variables are related by the homomorphisms

$$\begin{aligned} \rho_{N,M} : \kappa \Lambda_N &\rightarrow \kappa \Lambda_M \\ x_i &\mapsto x_i \quad i \leq M \\ x_i &\mapsto 0 \quad i > M, \end{aligned}$$

where $N > M$. The ring of symmetric polynomials in a countably infinite number variables is given by the projective limit

$$\kappa \Lambda = \varprojlim_N \kappa \Lambda_N,$$

relative to the homomorphisms $\rho_{N,M}$.

A convenient way of parametrising symmetric polynomials, is by partitions of integers.

Definition 3.10. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly descending sequence of non-negative integers. We refer to

$$|\lambda| = \sum_i \lambda_i$$

as the degree of λ and to

$$\ell(\lambda) = \#\{\lambda \neq 0\}$$

as the length of λ .

To each partition λ we associate a Young diagram, that is a collection of left aligned rows of boxes where the i th row consists of λ_i boxes. The boxes of a diagram are labelled by two integers (i, j) , where i labels the row and j the column. For every partition λ there is also conjugate partition λ' which is obtained by exchanging rows and columns in the Young diagram. For example the conjugate of the partition $(4, 2)$ is $(2, 2, 1, 1)$. For a box $s = (i, j)$ in a Young diagram let

$$\begin{aligned} a_\lambda(s) &= \lambda_i - j & a'_\lambda(s) &= j - 1 \\ \ell_\lambda(s) &= \lambda'_j - i & \ell'_\lambda(s) &= i - 1. \end{aligned}$$

Partitions admit a partial ordering \geq called the dominance ordering. For two partitions λ, μ of equal degree, $\lambda \geq \mu$ if and only if

$$\sum_{i=1}^n \lambda_i \geq \sum_{i=1}^n \mu_i, \quad n \geq 1.$$

Two examples of bases of ${}_{\kappa}\Lambda$ parametrised by partitions are the power sums

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

and the symmetric monomials

$$m_\lambda = \sum_{\sigma} \sum_{i \geq 1} x_{\sigma(i)}^{\lambda_i},$$

where the first sum runs over all distinct permutations of the entries of the partition λ . In the case of symmetric polynomials in N variables one must restrict oneself to partitions of length at most N , since

$$\rho_N(m_\lambda) = 0, \quad \text{for } \ell(\lambda) > N.$$

Definition 3.11. Let $(\ , \)_\kappa$ be the inner product of symmetric polynomials defined by

$$\begin{aligned} (\ , \)_\kappa : {}_\kappa\Lambda \otimes {}_\kappa\Lambda &\rightarrow \mathcal{K} \\ (p_\lambda, p_\mu)_\kappa &\mapsto \delta_{\lambda,\mu} z_\lambda \kappa^{\ell(\lambda)}, \end{aligned}$$

where

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

Proposition 3.12. For $\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O}$ such that $\kappa_0 \in \mathcal{O} \setminus \mathbb{Q}_{\leq 0}$ and $\kappa_1 \neq 0$, the symmetric polynomials ${}_\kappa\Lambda$ admit a basis of polynomials called Jack polynomials $P_\lambda(x; \kappa)$ that satisfy

1. $(P_\lambda(x; \kappa), P_\mu(x; \kappa))_\kappa = 0$ if $\lambda \neq \mu$
2. $P_\lambda(x; \kappa) = \sum_{\lambda \geq \mu} u_{\lambda,\mu}(\kappa) m_\mu$, where $u_{\lambda,\lambda}(\kappa) = 1$, $u_{\lambda,\mu}(\kappa) \in \mathcal{O}$ and the partial ordering \geq is the dominance ordering.

Definition 3.13. Let

$$b_\lambda(\kappa) = ((P_\lambda(x; \kappa), P_\lambda(x; \kappa))_\kappa)^{-1},$$

then the polynomials

$$Q_\lambda(x; \kappa) = b_\lambda(\kappa) P_\lambda(x; \kappa),$$

form a dual basis to $P_\lambda(x; \kappa)$, such that,

$$(P_\lambda(x; \kappa), Q_\mu(x; \kappa))_\kappa = \delta_{\lambda,\mu}.$$

Jack polynomials in an infinite number of variables satisfy some remarkable properties which we summarise in the following proposition.

Proposition 3.14. 1. For $\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O}$ such that $\kappa_0 \in \mathcal{O} \setminus \mathbb{Q}_{\leq 0}$ and $\kappa_1 \neq 0$, the coefficients $b_\lambda(\kappa)$ are given by

$$b_\lambda(\kappa) = ((P_\lambda(x; \kappa), P_\lambda(x; \kappa))_\kappa)^{-1} = \prod_{s \in \lambda} \frac{\kappa a(s) + \ell(s) + 1}{\kappa a(s) + \ell(s) + \kappa}$$

and are units of \mathcal{O} , that is, $\lim_{\varepsilon \rightarrow 0} b_\lambda(\kappa) \in \mathbb{C} \setminus \{0\}$.

2.

$$\prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\kappa} = \prod_{k \geq 1} e^{\frac{1}{\kappa} \frac{p_k(x) p_k(y)}{k}} = \sum_{\lambda} P_\lambda(x; \kappa) Q_\lambda(y; \kappa)$$

3. Let ω_β be the \mathcal{K} algebra endomorphism of ${}_{\mathcal{K}}\Lambda$ given by

$$\omega_\beta(p_r) = (-1)^{r-1} \beta p_r,$$

then the Jack polynomials satisfy

$$\omega_\kappa(P_\lambda(x; \kappa)) = Q_{\lambda'}(x; \kappa^{-1}).$$

4. Let ε_X be the \mathcal{K} -algebra homomorphism defined by

$$\begin{aligned} \varepsilon_X : {}_{\mathcal{K}}\Lambda &\rightarrow \mathcal{K} \\ p_r &\mapsto X \end{aligned}$$

for any $X \in \mathcal{K}$, then the Jack polynomials satisfy

$$\varepsilon_X(P_\lambda(z; \kappa)) = \prod_{s \in \lambda} \frac{X + \kappa a'(s) - \ell'(s)}{\kappa a(s) + \ell(s) + 1}$$

We return to symmetric polynomials in a finite number of variables.

Proposition 3.15. 1. A partition λ defines a non-zero Jack polynomial $P_\lambda(x; \kappa) \in {}_{\mathcal{K}}\Lambda_N$ if and only if $\ell(\lambda) \leq N$.

2. For the partition $\lambda = (m)^N$, that is a partition consisting of N copies of an integer m , the Jack polynomial $P_\lambda(x; \kappa) \in {}_{\mathcal{K}}\Lambda_N$ is given by

$$P_\lambda(x; \kappa) = \prod_{i=1}^N x_i^m.$$

Definition 3.16. For the symmetric polynomials in a finite number of variables, let $(\ , \)_\kappa^N$ be the inner product

$$(\ , \)_\kappa^N : {}_{\mathcal{K}}\Lambda^N \otimes_{\mathcal{K}} {}_{\mathcal{K}}\Lambda^N \rightarrow \mathcal{K}$$

such that

$$(f, g)_\kappa^N = \int_{[\Gamma_N(\kappa)]} \Delta_N(x; \kappa) \bar{f} g \prod_{i=1}^N \frac{dx_i}{x_i},$$

where $\bar{f} = f(x_1^{-1}, \dots, x_N^{-1})$.

Proposition 3.17. For two partitions λ, μ with $\ell(\lambda), \ell(\mu) \leq N$

$$\begin{aligned} (P_\lambda(x; \kappa), P_\mu(x; \kappa))_\kappa^N = \\ \delta_{\lambda, \mu} \prod_{s \in \lambda} \frac{(\kappa a(s) + \ell(s) + \kappa)(N + a'(s) - \ell'(s))}{(\kappa a(s) + \ell(s) + 1)(N + (a'(s) + 1)\kappa - \ell'(s) - 1)} \end{aligned}$$

As a direct consequence of the above proposition we have the corollary

Corollary 3.18. *For $\kappa = \kappa_0 + \kappa_1\varepsilon + \cdots \in \mathcal{O}$ such that $\kappa_0 \in \mathcal{O} \setminus \mathbb{Q}_{\leq 0}$ and $\kappa_1 \neq 0$*

1. *the inner product $(\ , \)_\kappa$ also defines an inner product on ${}_{\mathcal{O}}\Lambda$*

$${}_{\mathcal{O}}\Lambda \otimes_{\mathcal{O}} {}_{\mathcal{O}}\Lambda \rightarrow \mathcal{O},$$

2. *the map*

$$\int_{[\Gamma_N(\kappa)]} : {}_{\mathcal{K}}H^N(X_n, {}_{\mathcal{K}}\mathcal{L}(\kappa))^{S_N-} \rightarrow \mathcal{K}$$

preserves the \mathcal{O} lattice, that is

$$\int_{[\Gamma_N(\kappa)]} : {}_{\mathcal{O}}H^N(X_n, {}_{\mathcal{O}}\mathcal{L}(\kappa))^{S_N-} \rightarrow \mathcal{O},$$

where S_N- denotes the elements of the cohomology groups that are skew-symmetric with respect to the action of S_N .

3.4 Integrating on the \mathcal{O} -lattice

By Corollary 3.18 we have the following proposition

Proposition 3.19. *1. For $r \geq 1, s \in \mathbb{Z}$ the integration of the product of screening operators $\prod_{i=1}^r S_+(z_i)$ over the cycle $[\Gamma_r(\kappa_+(\varepsilon))]$ closes on the \mathcal{O} lattice, that is*

$$S_+^{[r]} = \int_{[\Gamma_N(\kappa_+(\varepsilon))]} \prod_{i=1}^r S_+(x_i) \prod_{i=1}^r dx_i : {}_{\mathcal{O}}F_{r,s} \rightarrow {}_{\mathcal{O}}F_{-r,s}.$$

2. *For $r \in \mathbb{Z}, s \geq 1$ the integration of the product of screening operators $\prod_{i=1}^s S_-(z_i)$ over the cycle $[\Gamma_s(\kappa_-(\varepsilon))]$ closes on the \mathcal{O} lattice, that is*

$$S_-^{[s]} = \int_{[\Gamma_N(\kappa_-(\varepsilon))]} \prod_{i=1}^s S_-(x_i) \prod_{i=1}^s dx_i : {}_{\mathcal{O}}F_{r,s} \rightarrow {}_{\mathcal{O}}F_{r,-s}.$$

Let

$${}_{\mathcal{O}}K_{1,1} = \ker(S_+ : {}_{\mathcal{O}}F_{1,1} \rightarrow {}_{\mathcal{O}}F_{-1,1}) \cap \ker(S_- : {}_{\mathcal{O}}F_{1,1} \rightarrow {}_{\mathcal{O}}F_{1,-1}),$$

then $({}_{\mathcal{O}}K_{1,1}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$ carries the structure of a VOA over \mathcal{O} and is an \mathcal{O} -lattice of the VOA $({}_{\mathcal{K}}K_{1,1}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$.

Remark 3.20. *It is known that*

$$({}_\kappa K_{1,1}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y) \cong {}_\kappa \text{Vir}_{c_{p_+, p_-}(\varepsilon)}$$

as a VOA, however, as we will show later

$${}_o K_{1,1} \otimes_{{}_o} \mathbb{C} = K_{1,1;0} \supsetneq \text{Vir}_{c_{p_+, p_-}}.$$

For each $h \in \mathcal{K}$, let ${}_\kappa M(h)$ be the ${}_\kappa U(\mathcal{L})$ -Verma module with highest weight h . Then the following is well known.

Proposition 3.21. 1. *The Verma module ${}_\kappa M(h)$ is not irreducible as a ${}_\kappa U(\mathcal{L})$ module if and only if $h = h_{r,s}(\varepsilon)$ for some $r \geq 1, s \geq 1$.*
 2. *For each $\beta \in \mathcal{K}$, consider the left ${}_\kappa U(\mathcal{L})$ -module ${}_\kappa F_\beta$, then there is a canonical ${}_\kappa U(\mathcal{L})$ -module map*

$$\begin{aligned} {}_\kappa M_{h_\beta} &\rightarrow {}_\kappa F_\beta \\ u_{h_\beta} &\mapsto |\beta\rangle, \end{aligned}$$

where u_{h_β} is the highest weight state of ${}_\kappa M_{h_\beta}$. This map is not an isomorphism if and only if $\beta = \beta_{r,s}$ or $\beta = \beta_{-r,-s}$ for some $r \geq 1, s \geq 1$.

Proposition 3.22. *Let $r \geq 1, s \geq 1$.*

1. *The sequences*

$$\begin{aligned} 0 \rightarrow {}_\kappa L(h_{r,s}(\varepsilon)) &\rightarrow {}_\kappa F_{r,s} \xrightarrow{S_+^{[r]}} {}_\kappa F_{-r,s} \rightarrow 0 \\ 0 \rightarrow {}_\kappa L(h_{r,s}(\varepsilon)) &\rightarrow {}_\kappa F_{r,s} \xrightarrow{S_-^{[s]}} {}_\kappa F_{r,-s} \rightarrow 0 \\ 0 \rightarrow {}_\kappa F_{r,-s} &\xrightarrow{S_+^{[r]}} {}_\kappa F_{-r,-s} \rightarrow {}_\kappa L(h_{-r,-s}(\varepsilon)) \rightarrow 0 \\ 0 \rightarrow {}_\kappa F_{-r,s} &\xrightarrow{S_-^{[s]}} {}_\kappa F_{-r,-s} \rightarrow {}_\kappa L(h_{-r,-s}(\varepsilon)) \rightarrow 0 \end{aligned}$$

are exact as ${}_\kappa U(L)$ -modules.

2. *The diagram*

$$\begin{array}{ccc} {}_\kappa F_{r,s} & \xrightarrow{S_+^{[r]}} & {}_\kappa F_{-r,s} \\ S_-^{[s]} \downarrow & & \downarrow S_-^{[s]} \\ {}_\kappa F_{r,-s} & \xrightarrow{S_+^{[r]}} & {}_\kappa F_{-r,-s} \end{array}$$

commutes.

3. The maps

$$\begin{aligned}\kappa M(h_{r,s}(\varepsilon)) &\rightarrow \kappa F_{-r,-s} \\ \kappa F_{r,s} = \kappa F_{-r,-s}^* &\rightarrow \kappa M(h_{r,s}(\varepsilon))^*\end{aligned}$$

are isomorphisms of left $\kappa U(\mathcal{L})$ -modules.

4. The $\kappa U(\mathcal{L})$ -module maps in 3 preserve the \mathcal{O} -lattices

$$\begin{aligned}0 \rightarrow \mathcal{O} M(h_{r,s}(\varepsilon)) &\longrightarrow \mathcal{O} F_{-r,-s} \\ 0 \rightarrow \mathcal{O} F_{r,s} &\longrightarrow \mathcal{O} M(h_{r,s}(\varepsilon))^*,\end{aligned}$$

where

$$\begin{aligned}\mathcal{O} M(h_{r,s}(\varepsilon)) &= \mathcal{O} U(\mathcal{L}) u_{h_{r,s}(\varepsilon)} \subset \kappa M(h_{r,s}(\varepsilon)) \\ \mathcal{O} M(h_{r,s}(\varepsilon))^* &= \bigoplus_{d \geq 0} \text{Hom}_{\mathcal{O}}(\mathcal{O} M(h_{r,s}(\varepsilon)) [h_{r,s}(\varepsilon) + d], \mathcal{O}).\end{aligned}$$

5. For each $r \geq 1, s \geq 1$ there exists a unique element $S_{r,s}(\kappa) \in \mathcal{O} U(\mathcal{L}_{<0})$ such that

$$S_{r,s}(\kappa) = (L_{-1})^{rs} + \dots$$

and

$$\sigma_{r,s}(\kappa) = S_{r,s}(\kappa) u_{h_{r,s}(\varepsilon)} \in \mathcal{O} M(h_{r,s}(\varepsilon))$$

satisfies

$$\begin{aligned}L_n \sigma_{r,s}(\kappa) &= 0, \quad n \geq 1 \\ L_0 \sigma_{r,s}(\kappa) &= (h_{r,s}(\varepsilon) + rs) \sigma_{r,s}(\kappa) \\ h_{r,s}(\varepsilon) + rs &= h_{r,-s}(\varepsilon) = h_{-r,s}(\varepsilon).\end{aligned}$$

6. The conformal weight $h_{r,s}(\varepsilon) + rs = h_{r,-s}(\varepsilon) = h_{-r,s}(\varepsilon)$ is precisely the conformal weight of $\sigma_{r,s}(\kappa)$, $S_+^{[r]} |\beta_{r,-s}(\varepsilon)\rangle$ and $S_+^{[s]} |\beta_{-r,s}(\varepsilon)\rangle$. Under the identification

$$\kappa M(h_{r,s}) \cong \kappa F_{-r,-s}$$

these three states are proportional to each other.

Propositions 3.21 and 3.22 are due to Feigin and Fuchs [FFa, FFb, FFc].

Since the above maps $S_+^{[r]}$, $S_-^{[s]}$ close on the \mathcal{O} lattices, we can take the limit of $\varepsilon \rightarrow 0$ and obtain integrals of products of screening operators over \mathbb{C} , which we will also denote by $S_+^{[r]}$, $S_-^{[s]}$. These new screening operators define maps between Fock modules F_β , $\beta \in X$.

For each $\gamma \in \mathbb{C} \setminus \{0\}$, let ρ_γ be the \mathbb{C} algebra isomorphism

$$\begin{aligned} \rho_\gamma : \Lambda &\rightarrow U(\mathfrak{b}_-) \\ p_n(x) &\mapsto \gamma b_{-n}, \quad n = 1, 2, \dots \end{aligned}$$

Proposition 3.23. *The action of the screening operators on $|\beta_{r,s}\rangle$ is given by*

1. For $r \geq 1$ and $s \in \mathbb{Z}$

$$S_+^{[r]} : F_{r,s} \rightarrow F_{-r,s},$$

such that

$$S_+^{[r]}|\beta_{r,s}\rangle = \begin{cases} 0 & s \geq 1 \\ \rho_{\frac{2}{\alpha_+}}(Q_{(s)^r}(x; \kappa_-))|\beta_{r,-s}\rangle & s \leq 0 \end{cases},$$

where $(s)^r$ is the partition consisting of r copies of s .

2. For $r \in \mathbb{Z}$ and $s \geq 1$

$$S_-^{[s]} : F_{r,s} \rightarrow F_{r,-s},$$

such that

$$S_-^{[s]}|\beta_{r,s}\rangle = \begin{cases} 0 & r \geq 1 \\ \rho_{\frac{2}{\alpha_-}}(Q_{(r)^s}(x; \kappa_+))|\beta_{r,-s}\rangle & s \leq 0 \end{cases},$$

where $(r)^s$ is the partition consisting of s copies of r .

3. For $r, s \geq 1$

$$S_+^{[r]}|\beta_{r,-s}\rangle = (-1)^{rs} b_{(s)^r}(\kappa_-) S_-^{[s]}|\beta_{-r,s}\rangle.$$

4 The structure of Virasoro modules at central charge c_{p_+, p_-}

The way in which Fock spaces decompose into Virasoro modules was determined by Feigin and Fuchs in [FFa, FFb, FFc]. For a more modern and detailed account see [IK]. In this section we will see how Fock-modules decompose as Virasoro modules, calculate the kernels and images of screening operators mapping between Fock-modules and introduce infinite sums of kernels and images that will later turn out to be \mathcal{M}_{p_+, p_-} -modules.

Let

$$\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} c,$$

be the Virasoro algebra, where c is the central element. Furthermore let

$$c_{p_+, p_-} = 1 - 6 \frac{(p_+ - p_-)^2}{p_+ p_-}$$

and let $U(\mathcal{L})$ be the universal enveloping algebra of \mathcal{L} with $c = c_{p_+, p_-} \text{id}$. The Virasoro vertex operator algebra Vir_{p_+, p_-} is given by the restriction of \mathcal{F}_{p_+, p_-} to the subVOA $\text{Vir}_{p_+, p_-} = (U(\mathcal{L})|0\rangle, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0 b_2)|0\rangle, Y)$.

4.1 The category $U(\mathcal{L})\text{-mod}$ of Virasoro modules at central charge c_{p_+, p_-}

The purpose of this subsection is to give a description of the category $U(\mathcal{L})\text{-mod}$ and to give socle sequence decompositions of Fock-modules in terms of simple Virasoro modules.

Definition 4.1. *Let*

$$H = \{h_\beta | \beta \in X\}$$

be the set of highest conformal weights. Two non-equal Heisenberg weights $\beta, \beta' \in X$ correspond to the same conformal weight if and only if $\beta' = \alpha_0 - \beta$.

Definition 4.2. 1. *For $1 \leq r < p_+$, $1 \leq s < p_-$ let*

$$\Delta_{r,s} = \Delta_{p_+ - r, p_- - s} = h_{r,s;0}.$$

2. *The Kac table \mathcal{T} is the quotient set*

$$\mathcal{T} = \{(r, s) | 1 \leq r < p_+, 1 \leq s < p_-\} / \sim,$$

where $(r, s) \sim (r', s')$ if and only if $r' = p_+ - r, s' = p_- - s$. The Kac table is the set of all classes $[(r, s)]$ such that the conformal weights $\Delta_{r,s}$ are distinct.

3. For $1 \leq r \leq p_+, 1 \leq s \leq p_-, n \geq 0$ let

$$\Delta_{r,s;n}^+ = \begin{cases} h_{p_+,p_-;-2n} & r = p_+, s = p_- \\ h_{p_+-r,p_-;-2n-1} & r \neq p_+, s = p_- \\ h_{p_+,p_- - s;2n+1} & r = p_+, s \neq p_- \\ h_{p_+-r,s;-2n-1} & r \neq p_+, s \neq p_- \end{cases},$$

$$\Delta_{r,s;n}^- = \begin{cases} h_{p_+,p_-;-2n-1} & r = p_+, s = p_- \\ h_{p_+-r,p_-;-2n-2} & r \neq p_+, s = p_- \\ h_{p_+,p_- - s;2n+2} & r = p_+, s \neq p_- \\ h_{p_+-r,s;-2n-2} & r \neq p_+, s \neq p_- \end{cases}.$$

Definition 4.3. Let $U(\mathcal{L})\text{-Mod}$ be the abelian category whose objects are left Virasoro modules and whose morphisms are Virasoro-homomorphisms, such that

1. The central charge c is

$$c_{p_+,p_-} = 1 - 6 \frac{(p_+ - p_-)^2}{p_+ p_-}.$$

2. Every object M decomposes into a direct sum of generalised L_0 eigenspaces

$$M = \bigoplus_{h \in \mathbb{C}} M[h]$$

$$M[h] = \{u \in M \mid \exists n \geq 1, \text{ s.t. } (L_0 - h)^n u = 0\}$$

where $\dim M[h] < \infty$. For all $h \in \mathbb{C}$ and there are only a countable number of h for which $M[h]$ is non-trivial.

3. There exists an anti-involution $\sigma : U(\mathcal{L}) \rightarrow U(\mathcal{L})$ such that $\sigma(L_n) = L_{-n}, \in \mathbb{Z}$.

4. For every object $M \in U(\mathcal{L})\text{-mod}$, there exists the contragredient object M^*

$$M^* = \bigoplus_{h \in \mathbb{C}} \text{Hom}(M[h], \mathbb{C}),$$

on which σ induces the structure of a left $U(\mathcal{L})$ -module by

$$\langle L_n \varphi, u \rangle = \langle \varphi, \sigma(L_n) u \rangle, \quad \varphi \in M^*, u \in M.$$

Note that $(M^*)^* \cong M$.

Examples of objects of $U(\mathcal{L})$ -mod are

1. Verma modules $M(h)$, that is modules generated by a state u_h satisfying

$$\begin{aligned} L_n u_h &= 0, \quad n \geq 1 \\ L_0 u_h &= h u_h \end{aligned}$$

such that

$$\begin{aligned} U(\mathcal{L}_-) &\rightarrow M(h) \\ P &\mapsto P u_h \end{aligned}$$

is an isomorphism of vector spaces, where $U(\mathcal{L}_-)$ is the universal enveloping algebra of the Virasoro generators with negative mode number.

2. Simple modules $L(h)$, the simple quotients of the Verma modules $M(h)$. Note that $L(h)$ is isomorphic to $M(h)$, for $h \notin \mathbb{C} \setminus H$.
3. Fock modules F_β for $\beta \in \mathbb{C}$. Note that F_β is isomorphic to $M(h_\beta)$, for $\beta \notin \mathbb{C} \setminus X$.

Definition 4.4. A socle series of a Virasoro module M , is a series of semi-simple modules $S_i(M)$, defined in the following way. Let

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

be an ascending series of submodules of M , such that $S_i(M) = M_i/M_{i-1}$, $i \geq 1$ is the maximal semi-simple submodule of M_{i+1}/M_{i-1} , that is, M_i is the largest submodule of M such that M_i/M_{i-1} is semi-simple. The $S_i(M)$ are called the components of M . If there exists an element M_n of the filtration, such that, $M_n = M$ and $M_{n-1} \neq M$, then we say that the socle series has length n .

Definition 4.5. We define $U(\mathcal{L})$ -mod to be the full subcategory of $U(\mathcal{L})$ -Mod such that for all objects M in $U(\mathcal{L})$ -mod

1. the socle series have finite length,
2. the conformal weights h of the simple modules $L(h)$, appearing in the components of M , are elements of H .

Proposition 4.6. 1. For each $\beta \in X$ the Fock module F_β is an object of $U(\mathcal{L})$ -mod.

2. There are four cases of socle series for the Fock modules $F_{r,s;n}$

(I) For $1 \leq r < p_+$, $1 \leq s < p_-$, $n \in \mathbb{Z}$,

$$\begin{aligned} S_1(F_{r,s;n}) &= \bigoplus_{k \geq 0} \mathcal{L}(h_{r,p_- - s; |n| + 2k + 1}), \\ S_2(F_{r,s;n}) &= \bigoplus_{k \geq a} \mathcal{L}(h_{r,s; |n| + 2k}) \\ &\quad \oplus \bigoplus_{k \geq 1-a} \mathcal{L}(h_{p_+ - r, p_- - s; |n| + 2k}), \\ S_3(F_{r,s;n}) &= \bigoplus_{k \geq 0} \mathcal{L}(h_{p_+ - r, s; |n| + 2k + 1}), \end{aligned}$$

where $a = 0$ if $n \geq 0$ and $a = 1$ if $n < 0$.

(II₊) For $1 \leq s < p_-$, $n \in \mathbb{Z}$,

$$\begin{aligned} S_1(F_{p_+,s;n}) &= \bigoplus_{k \geq 0} \mathcal{L}(h_{p_+, p_- - s; |n| + 2k + 1}), \\ S_2(F_{p_+,s;n}) &= \bigoplus_{k \geq a} \mathcal{L}(h_{p_+, s; |n| + 2k}), \end{aligned}$$

where $a = 0$ if $n \geq 1$ and $a = 1$ if $n < 1$.

(II₋) For $1 \leq r < p_+$, $n \in \mathbb{Z}$,

$$\begin{aligned} S_1(F_{r,p_-;n}) &= \bigoplus_{k \geq 0} \mathcal{L}(h_{r,p_-; |n| + 2k}), \\ S_2(F_{r,p_-;n}) &= \bigoplus_{k \geq a} \mathcal{L}(h_{p_+ - r, p_-; |n| + 2k - 1}), \end{aligned}$$

where $a = 1$ if $n \geq 1$ and $a = 0$ if $n < 1$.

(III) For $n \in \mathbb{Z}$, the Fock space $F_{p_+,p_-;n}$ is semi-simple as a Virasoro module

$$S_1(F_{p_+,p_-;n}) = F_{p_+,p_-;n} = \bigoplus_{k \geq 0} \mathcal{L}(h_{p_+,p_-; |n| + 2k}).$$

Figures 1 and 2 visualise the socle sequence decomposition of Fock modules.

(I) $F_{r,s;n}$

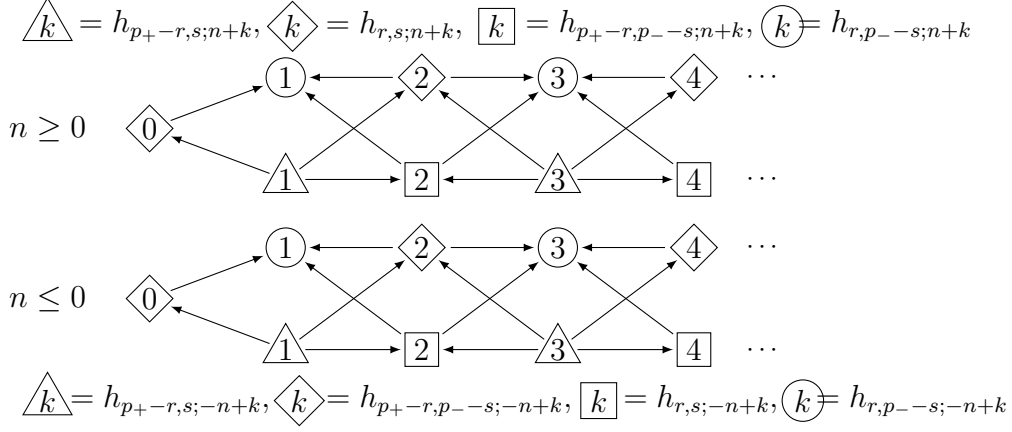
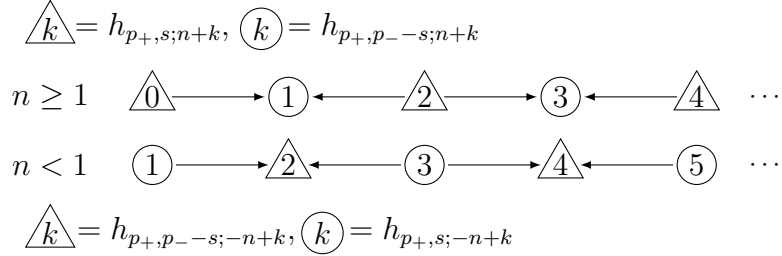


Figure 1: The socle sequence of $F_{r,s;n}$ for $1 \leq r < p_+$, $1 \leq s < p_-$. The circles correspond to S_1 the squares and diamonds to S_2 and the triangles to S_3 . The arrows indicate that the Virasoro algebra generates the tip of the arrow by acting on the base.

(II₋) $F_{p_+,s;n}$



(II₊) $F_{r,p_-;n}$

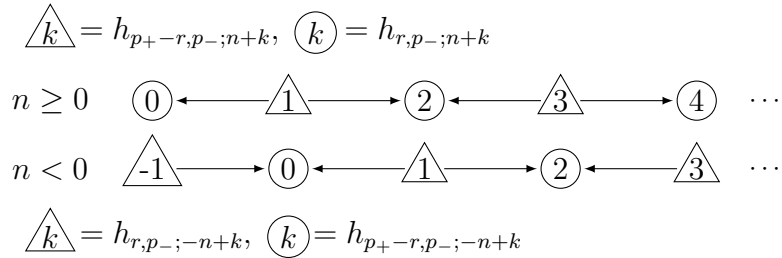


Figure 2: The socle sequence for $F_{r,s;n}$ when either $r = p_+$ or $s = p_-$. The circles correspond to S_1 and the triangles to S_2 . The arrows indicate that the Virasoro algebra generates the tip of the arrow by acting on the base.

4.2 Kernels and images of the screening operators $S_+^{[r]}$, $S_-^{[s]}$

In this subsection we give the socle sequences of the kernels and images of the screening operators $S_+^{[r]}$, $S_-^{[s]}$.

Definition 4.7. We denote the kernels and images of screening operators $S_+^{[r]}$, $S_-^{[s]}$ by

1. For $1 \leq r < p_+$, $1 \leq s \leq p_-$, $n \in \mathbb{Z}$

$$\begin{aligned} K_{r,s;n;+} &= \ker S_+^{[r]} : F_{r,s;n} \rightarrow F_{p_+-r,s;n+1} \\ X_{p_+-r,s;n+1;+} &= \operatorname{im} S_+^{[r]} : F_{r,s;n} \rightarrow F_{p_+-r,s;n+1} \end{aligned}$$

2. For $1 \leq r \leq p_+$, $1 \leq s < p_-$, $n \in \mathbb{Z}$

$$\begin{aligned} K_{r,s;n;-} &= \ker S_-^{[s]} : F_{r,s;n} \rightarrow F_{r,p_--s;n-1} \\ X_{r,p_--s;n-1;-} &= \operatorname{im} S_-^{[s]} : F_{r,s;n} \rightarrow F_{r,p_--s;n-1} \end{aligned}$$

3. For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, $n \in \mathbb{Z}$

$$\begin{aligned} K_{r,s;n} &= K_{r,s;n;+} \cap K_{r,s;n;-} & X_{r,s;n} &= X_{r,s;n;+} \cap X_{r,s;n;-} \\ K_{r,p_--;n} &= K_{r,p_--;n;+} & K_{p_+,s;n} &= K_{p_+,s;n;-} \end{aligned}$$

Proposition 4.8. For $1 \leq r < p_+$, $1 \leq s < p_-$ the socle sequences of the kernels and images of the screening operators S_+ and S_- are given by

$$\begin{aligned} S_1(K_{r,s;n;+}) &= \bigoplus_{k \geq 1} L(h_{r,p_--s;n+2k-1}), \\ S_2(K_{r,s;n;+}) &= \bigoplus_{k \geq 1} L(h_{r,s;n+2(k-1)}), \\ S_1(X_{r,s;n+1;+}) &= \bigoplus_{k \geq 1} L(h_{r,p_--s;n+2k}), \\ S_2(X_{r,s;n+1;+}) &= \bigoplus_{k \geq 1} L(h_{r,s;n+2k-1}), \end{aligned}$$

for $n \geq 0$,

$$\begin{aligned}
S_1(K_{r,s;n;+}) &= \bigoplus_{k \geq 1} L(h_{r,p_- - s; -n+2k-1}) , \\
S_2(K_{r,s;n;+}) &= \bigoplus_{k \geq 1} L(h_{r,s; -n+2k}) , \\
S_1(X_{r,s;n+1;+}) &= \bigoplus_{k \geq 1} L(h_{r,p_- - s; -n+2(k-1)}) , \\
S_2(X_{r,s;n+1;+}) &= \bigoplus_{k \geq 1} L(h_{r,s; -n+2k-1}) ,
\end{aligned}$$

for $n \leq -1$,

$$\begin{aligned}
S_1(K_{r,s;n;-}) &= \bigoplus_{k \geq 1} L(h_{r,p_- - s; n+2k-1}) , \\
S_2(K_{r,s;n;-}) &= \bigoplus_{k \geq 1} L(h_{p_+ - r, p_- - s; n+2k}) , \\
S_1(X_{r,s;n+1;-}) &= \bigoplus_{k \geq 1} L(h_{r,p_- - s; n+2(k-1)}) , \\
S_2(X_{r,s;n+1;-}) &= \bigoplus_{k \geq 1} L(h_{p_+ - r, p_- - s; n+2k-1}) ,
\end{aligned}$$

for $n \geq 1$,

$$\begin{aligned}
S_1(K_{r,s;n;-}) &= \bigoplus_{k \geq 1} L(h_{r,p_- - s; -n+2k-1}) , \\
S_2(K_{r,s;n;-}) &= \bigoplus_{k \geq 1} L(h_{p_+ - r, p_- - s; -n+2(k-1)}) , \\
S_1(X_{r,s;n+1;-}) &= \bigoplus_{k \geq 1} L(h_{r,p_- - s; -n+2k}) , \\
S_2(X_{r,s;n+1;-}) &= \bigoplus_{k \geq 1} L(h_{p_+ - r, p_- - s; -n+2k-1}) .
\end{aligned}$$

for $n \leq 0$. For $r = p_+$ or $s = p_-$ the kernels and images are semisimple and we have

$$K_{r,p_-;n;+} = X_{r,p_-;n;+} = S_1(F_{r,p_-;n}) \quad K_{p_+,s;n;-} = X_{p_+,s;n;-} = S_1(F_{p_+,s;n}) .$$

The following proposition is due to Felder [Fel].

Proposition 4.9. 1. For $1 \leq r < p_+$, $1 \leq s \leq p_-$ and $n \in \mathbb{Z}$ the screening operator S_+ defines the sequence

$$\cdots \xrightarrow{S_+^{[r]}} F_{p_+-r,s;n-1} \xrightarrow{S_+^{[p_+-r]}} F_{r,s;n} \xrightarrow{S_+^{[r]}} F_{p_+-r,s;n+1} \xrightarrow{S_+^{[p_+-r]}} \cdots$$

This sequence is exact for $s = p_-$. For $1 \leq s < p_-$ it is exact everywhere except in $F_{r,s;0}$, where the cohomology is isomorphic to

$$K_{r,s;0;+}/X_{r,s;0;+} \cong L(h_{r,s;0}).$$

2. For $1 \leq r \leq p_+$, $1 \leq s < p_-$ and $n \in \mathbb{Z}$ the screening operator S_- defines the sequence

$$\cdots \xrightarrow{S_-^{[s]}} F_{r,p_--s;n+1} \xrightarrow{S_-^{[p_--s]}} F_{r,s;n} \xrightarrow{S_-^{[s]}} F_{r,p_--s;n-1} \xrightarrow{S_-^{[p_--s]}} \cdots$$

This sequence is exact for $r = p_+$. For $1 \leq r < p_+$ it is exact everywhere except in $F_{r,s;0}$, where the cohomology is isomorphic to

$$K_{r,s;0;-}/X_{r,s;0;-} \cong L(h_{r,s;0}).$$

Theorem A. Let $1 \leq r < p_+$, $1 \leq s \leq p_-$, all singular vectors of the Fock modules $F_{r,s;n}$ can be expressed as the images of the screening operators S_+ and S_- .

1. The singular vectors at levels $h_{r,p_--s;|n|+2k-1}$, $k \geq 0$ are given by

$$S_+^{[(k+n+1)p_+-r]} |\beta_{p_+-r,s;-1-2k-n}\rangle \in F_{r,s;n}$$

for $n \geq 0$ and

$$S_+^{[(k+1)p_+-r]} |\beta_{p_+-r,s;-1-2k+n}\rangle \in F_{r,s;n}$$

for $n \leq 0$.

2. The singular vectors at levels $h_{r,p_--s;|n|+2k-1}$, $k \geq 0$ are given by

$$S_-^{[(k+1)p_--s]} |\beta_{r,p_--s;2k+n+1}\rangle \in F_{r,s;n}$$

for $n \geq 0$ and

$$S_-^{[(k-n+1)p_--s]} |\beta_{r,p_--s;2k-n+1}\rangle \in F_{r,s;n}$$

for $n \leq 0$.

A weaker version of this theorem, that holds for $S_+^{[r]}$, $r < p_+$ and $S_-^{[s]}$, $s < p_-$ or for the case when the monodromies of products of S_+ and S_- are not roots of unity, was already developed in [MY].

4.3 From $U(\mathcal{L})$ -mod to \mathcal{M}_{p_+, p_-} -mod

The purpose of this subsection is to define certain infinite direct sums of kernels and images of $S_+^{[s]}$, $S_-^{[s]}$ which will turn later out to be modules of \mathcal{M}_{p_+, p_-} .

Definition 4.10. Let $1 \leq r \leq p_+$, $1 \leq s \leq p_-$ and $n \geq 0$.

1. Let $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ be the direct sums of kernels and images

$$\begin{aligned}\mathcal{K}_{r,s}^+ &= \bigoplus_{n \in \mathbb{Z}} K_{r,s;2n} & \mathcal{X}_{r,s}^+ &= \bigoplus_{n \in \mathbb{Z}} X_{r,s;2n} \\ \mathcal{K}_{r,s}^- &= \bigoplus_{n \in \mathbb{Z}} K_{r,s;2n+1} & \mathcal{X}_{r,s}^- &= \bigoplus_{n \in \mathbb{Z}} X_{r,s;2n+1}\end{aligned}$$

2. Let $V_{r,s;n}^\pm$ be the spaces of singular vectors of conformal weight $\Delta_{r,s}^\pm$

$$V_{r,s;n}^\pm = \{u \in \mathcal{X}_{r,s}^\pm \mid L_0 u = \Delta_{r,s}^\pm u\}.$$

We call these spaces soliton sectors. The elements of $V_{r,s;n}^\pm$ are called soliton vectors.

Remark 4.11. By Proposition 3.23 there are two equivalent basis of $V_{r,s;n}^\pm$ one in terms of S_+

$$\begin{aligned}V_{r,s;n}^+ &= \begin{cases} \bigoplus_{m=-n}^n \mathbb{C} S_+^{[(n+m)p_+]} |\beta_{p_+, p_-; -2n}\rangle & r = p_+, s = p_- \\ \bigoplus_{m=-n}^n \mathbb{C} S_+^{[(n+m)p_+]} |\beta_{p_+, s; -2n}\rangle & r = p_+, s \neq p_- \\ \bigoplus_{m=-n}^n \mathbb{C} S_+^{[(n+m+1)p_+ - r]} |\beta_{p_+ - r, p_-; -2n-1}\rangle & r \neq p_+, s = p_- \\ \bigoplus_{m=-n}^n \mathbb{C} S_+^{[(n+m+1)p_+ - r]} |\beta_{p_+ - r, s; -2n-1}\rangle & r \neq p_+, s \neq p_- \end{cases} \\ V_{r,s;n}^- &= \begin{cases} \bigoplus_{m=-n}^{n+1} \mathbb{C} S_+^{[(n+m)p_+]} |\beta_{p_+, p_-; -2n-1}\rangle & r = p_+, s = p_- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C} S_+^{[(n+m)p_+]} |\beta_{p_+, s; -2n-1}\rangle & r = p_+, s \neq p_- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C} S_+^{[(n+m+1)p_+ - r]} |\beta_{p_+ - r, p_-; -2n-2}\rangle & r \neq p_+, s = p_- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C} S_+^{[(n+m+1)p_+ - r]} |\beta_{p_+ - r, s; -2n-2}\rangle & r \neq p_+, s \neq p_- \end{cases},\end{aligned}$$

the other in terms of S_-

$$\begin{aligned}V_{r,s;n}^+ &= \begin{cases} \bigoplus_{m=-n}^n \mathbb{C} S_-^{[(n-m)p_-]} |\beta_{p_+, p_-; 2n}\rangle & r = p_+, s = p_- \\ \bigoplus_{m=-n}^n \mathbb{C} S_-^{[(n-m+1)p_- - s]} |\beta_{p_+, p_- - s; 2n+1}\rangle & r = p_+, s \neq p_- \\ \bigoplus_{m=-n}^n \mathbb{C} S_-^{[(n-m)p_-]} |\beta_{r, p_-; 2n}\rangle & r \neq p_+, s = p_- \\ \bigoplus_{m=-n}^n \mathbb{C} S_-^{[(n-m+1)p_- - s]} |\beta_{r, p_- - s; 2n+1}\rangle & r \neq p_+, s \neq p_- \end{cases} \\ V_{r,s;n}^- &= \begin{cases} \bigoplus_{m=-n}^{n+1} \mathbb{C} S_-^{[(n-m+1)p_-]} |\beta_{p_+, p_-; 2n+1}\rangle & r = p_+, s = p_- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C} S_-^{[(n-m+2)p_- - s]} |\beta_{p_+, p_- - s; 2n+2}\rangle & r = p_+, s \neq p_- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C} S_-^{[(n-m+1)p_-]} |\beta_{r, p_-; 2n+1}\rangle & r \neq p_+, s = p_- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C} S_-^{[(n-m+2)p_- - s]} |\beta_{r, p_- - s; 2n+2}\rangle & r \neq p_+, s \neq p_- \end{cases}.\end{aligned}$$

Proposition 4.12. *As Virasoro modules the spaces $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ decompose as*

1. *For $1 \leq r < p_+$ and $1 \leq s < p_-$*

$$\begin{aligned}\mathcal{K}_{r,s}^+ &= U(\mathcal{L})|\beta_{r,s;0}\rangle \oplus \bigoplus_{n \geq 1} L(\Delta_{r,s;n}^+) \otimes V_{r,s;n}^+ \\ \mathcal{X}_{r,s}^+ &= \bigoplus_{n \geq 0} L(\Delta_{r,s;n}^+) \otimes V_{r,s;n}^+ \\ \mathcal{K}_{r,s}^- &= \mathcal{X}_{r,s}^- = \bigoplus_{n \geq 0} L(\Delta_{r,s;n}^-) \otimes V_{r,s;n}^-\end{aligned}$$

2. *For $r = p_+$ and $1 \leq s < p_-$*

$$\begin{aligned}\mathcal{K}_{p_+,s}^+ &= \mathcal{X}_{p_+,s}^+ = \bigoplus_{n \geq 0} L(\Delta_{p_+,s;n}^+) \otimes V_{p_+,s;n}^+ \\ \mathcal{K}_{p_+,s}^- &= \mathcal{X}_{p_+,s}^- = \bigoplus_{n \geq 0} L(\Delta_{p_+,s;n}^-) \otimes V_{p_+,s;n}^-\end{aligned}$$

3. *For $1 \leq r < p_+$ and $s = p_-$*

$$\begin{aligned}\mathcal{K}_{r,p_-}^+ &= \mathcal{X}_{r,p_-}^+ = \bigoplus_{n \geq 0} L(\Delta_{r,p_-;n}^+) \otimes V_{r,p_-;n}^+ \\ \mathcal{K}_{r,p_-}^- &= \mathcal{X}_{r,p_-}^- = \bigoplus_{n \geq 0} L(\Delta_{r,p_-;n}^-) \otimes V_{r,p_-;n}^-\end{aligned}$$

4. *For $r = p_+$ and $s = p_-$*

$$\begin{aligned}\mathcal{K}_{p_+,p_-}^+ &= \mathcal{X}_{p_+,p_-}^+ = \bigoplus_{n \geq 0} L(\Delta_{p_+,p_-;n}^+) \otimes V_{p_+,p_-;n}^+ \\ \mathcal{K}_{p_+,p_-}^- &= \mathcal{X}_{p_+,p_-}^- = \bigoplus_{n \geq 0} L(\Delta_{p_+,p_-;n}^-) \otimes V_{p_+,p_-;n}^-\end{aligned}$$

Proposition 4.13. *For $1 \leq r < p_+$, $1 \leq s < p_-$, the $\mathcal{K}_{r,s}^+$ satisfy the following exact sequence as Virasoro modules*

$$0 \longrightarrow \mathcal{X}_{r,s}^+ \longrightarrow \mathcal{K}_{r,s}^+ \longrightarrow L(h_{r,s;0}) \longrightarrow 0.$$

Definition 4.14. *For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$ let $E, F : \mathcal{K}_{r,s}^\pm \rightarrow \mathcal{K}_{r,s}^\pm$ be the Virasoro endomorphisms characterised by the following action on the soliton sectors $V_{r,s;n}^\pm$ of $\mathcal{K}_{r,s}^\pm$.*

1. On the basis in terms of the screening operator S_+ of Remark 4.11, the map E acts by increasing the power of S_+ by p_+ . For example $ES_+^{[p_+-1]}|\beta_{p_+-1,1;-3}\rangle = S_+^{[2p_+-1]}|\beta_{p_+-1,1;-3}\rangle$
2. On the basis in terms of the screening operator S_- of Remark 4.11, the map F acts by increasing the power of S_- by p_- . For example $FS_-^{[p_--1]}|\beta_{1,p_--1;-3}\rangle = S_-^{[2p_--1]}|\beta_{1,p_--1;-3}\rangle$

We call the Virasoro endomorphisms E, F and $H = [E, F]$ Frobenius maps.

5 The extended W -algebra \mathcal{M}_{p_+, p_-}

5.1 Frobenius currents

In this section the Frobenius maps $E, F, H \in \text{End}_{U(\mathcal{L})}(\mathcal{K}_{r,s}^\pm)$ of the previous section will be extended to operators $E(z), F(z), H(z) \in \text{End}_{\mathbb{C}}(\mathcal{K}_{r,s}^+)[[z, z^{-1}]]$

Theorem B. For $1 \leq r \leq p_+, 1 \leq s \leq p_-$ and $X \in \{E, F, H\}$ there exists a unique operator $X(z) \in \text{End}(\mathcal{K}_{r,s}^+)[[z, z^{-1}]]$ such that

1. $X(z) = \sum_n X[n]z^{-n-1}$,
2. $X[0] = X \in \text{End}_{U(\mathcal{L})}(\mathcal{K}_{r,s}^\pm)$,
3. $[L_n, X(z)] = z^n(z\frac{d}{dz} + (n+1))X(z)$,

that is, $X(z) \in \text{End}_{\mathbb{C}}(\mathcal{K}_{r,s}^\pm)[[z, z^{-1}]]$ is a primary field of conformal weight 1. We call $X(z)$ the Frobenius current associated to $X \in \text{End}_{U(\mathcal{L})}(\mathcal{K}_{r,s}^\pm)$.

Proof. We show the case $\mathcal{K}_{r,s}^\pm = \mathcal{K}_{1,1}^+ = \mathcal{M}_{p_+, p_-}$. The other cases follow in the same way. Recall that as a $U(\mathcal{L})$ -module $\mathcal{K}_{1,1}^+$ decomposes as

$$\mathcal{K}_{1,1}^+ = U(\mathcal{L})|0\rangle \oplus \bigoplus_{n \geq 1} U(\mathcal{L})V_n,$$

where $U(\mathcal{L})V_n = L(\Delta_n) \otimes V_n$. Furthermore the sequence

$$0 \rightarrow L(\Delta_0) \rightarrow U(\mathcal{L})|0\rangle \rightarrow L(0) \rightarrow 0$$

is exact as a sequence of $U(\mathcal{L})$ -modules. Since $X \in \{E, H, F\}$ is a $U(\mathcal{L})$ -module map and satisfies

$$\begin{aligned} X &= 0 \quad \text{on } U(\mathcal{L})|0\rangle \\ X &: U(\mathcal{L})V_n \rightarrow U(\mathcal{L})V_n, \end{aligned}$$

it is sufficient to show that for each $n \geq 1$ there exists an

$$X(z) \in \text{End}(U(\mathcal{L})V_n)[[z, z^{-1}]]$$

such that

$$X(z) = \sum_n X[n]z^{-n-1}$$

and $X[0] = X \in \text{End}_{U(\mathcal{L})}(U(\mathcal{L})V_n)$

$$[L_n, X(z)] = z^n \left(z \frac{d}{dz} + (n+1) \right) X(z).$$

On $U(\mathcal{L})|0\rangle$ we set $X(z) = 0$.

The existence of $X(z)$ can be proved by using the principles of conformal field theory laid out in [BPZ] and Theorem 3.2 of [FFa]. The details are left to the reader. \square

Theorem C. *Let $1 \leq r \leq p_+, 1 \leq s \leq p_-$. For each $X \in \{E, F, H\}$ and $A \in \mathcal{K}_{1,1}^+$, the fields $X(z), Y(A; z)$ are local to each other in $\text{End}_{\mathbb{C}}(\mathcal{K}_{r,s}^{\pm})[[z, z^{-1}]]$.*

Proof. At first choose $A \in \mathcal{K}_{1,1}^+$ to be a highest weight vector of the Virasoro algebra. Then A has conformal weight 0 or Δ_n for some $n \geq 0$ and both $X(z)$ and $Y(A; z)$ are primary fields with respect to the Virasoro algebra. It follows from the principles of conformal field theory that $X(z)$ and $Y(A; z)$ are local to both each other and to the energy momentum tensor $T(z)$ [BPZ]. By Dong's Lemma [FBZ] it then follows that $X(z)$ and $Y(A; z)$ are local to each other for all $A \in \mathcal{K}_{1,1}^+$. \square

Corollary 5.1. *1. The adjoint action of the the Frobenius maps E, F, H define derivations of \mathcal{M}_{p_+, p_-} , that is, for $X \in \{E, F, H\}$ and $A, B \in \mathcal{K}_{1,1}^+$*

$$\begin{aligned} [X, Y(A; z)] &= Y(XA; z), \\ [X, Y(A; z)Y(B; w)] &= [X, Y(A; z)]Y(B; w) + Y(A; z)[X, Y(B; w)]. \end{aligned}$$

2. The adjoint action of the the Frobenius maps E, F, H define derivations of the universal enveloping algebra $U(\mathcal{M}_{p_+, p_-})$, that is, for $X \in \{E, F, H\}$ and $P, Q \in U(\mathcal{M}_{p_+, p_-})$

$$[X, PQ] = [X, P]Q + P[X, Q]$$

and for $A \in \mathcal{K}_{1,1}^+, n \in \mathbb{Z}$

$$[X, A[n]] = (XA)[n].$$

Proof. Since $X(z)$, $X \in \{E, F, H\}$ is local, it follows that the commutator of X and $Y(A; z)$ is again a field of \mathcal{M}_{p_+, p_-} , that is, there exists a $B \in \mathcal{K}_{1,1}^+$ such that

$$[X, Y(A; z)] = Y(B; z).$$

This B is uniquely determined by the operator state correspondence

$$[X, Y(A; z)]|0\rangle = XY(A; z)|0\rangle = XA + z\mathcal{K}_{1,1}^+[[z]].$$

Therefore $B = XA$. □

5.2 The algebra structure of \mathcal{M}_{p_+, p_-}

After studying the screening operators S_+ and S_- in the previous sections, we now turn to the extended W -algebra \mathcal{M}_{p_+, p_-} .

Definition 5.2. *The extended W -algebra $\mathcal{M}_{p_+, p_-} = (\mathcal{K}_{1,1}^+, |0\rangle, T, Y)$ is a sub-VOA of \mathcal{V}_{p_+, p_-} , where the vacuum vector, conformal vector and vertex operator map are those of \mathcal{V}_{p_+, p_-} ; and*

$$\mathcal{K}_{1,1}^+ = \ker S_+ \cap \ker S_- \subset V_{1,1}^+.$$

We specialise some of the notation of Section 4.

Definition 5.3. 1. For $n \geq 0$ let

$$\begin{aligned}\alpha_n^+ &= \beta_{p_+-1, 1; -2n-1} = \alpha_+ - (n+1)\alpha, \\ \alpha_n^- &= \beta_{1, p_- - 1; 2n+1} = \alpha_- + (n+1)\alpha,\end{aligned}$$

such that $\alpha_n^- = \alpha_n^{+\vee}$. The conformal weight corresponding to these Heisenberg weights is

$$\Delta_{1,1;n}^+ = h_{\alpha_n^\pm} = ((n+1)p_+ - 1)((n+1)p_- - 1).$$

We will abbreviate $\Delta_{1,1;n}^+$ as Δ_n .

2. Let $\lambda_{n,m}^+$ and $\lambda_{n,m}^-$ be the conjugate partitions

$$\begin{aligned}\lambda_{n,m}^+ &= ((n-m+1)p_- - 1)^{(n+m+1)p_+ - 1} \\ \lambda_{n,m}^- &= ((n+m+1)p_+ - 1)^{(n-m+1)p_- - 1}.\end{aligned}$$

3. For $n \geq 0$ and $-n \leq m \leq n$ let $W_{n,m}$ be the basis of the soliton sector $V_{1,1;n}^+$ given by

$$W_{n,m} = S_+^{[(n+m+1)p_+-1]} |\alpha_n^+\rangle,$$

such that for $n \geq 0$

$$\begin{aligned} EW_{n,n} &= 0 & EW_{n,m} &= W_{n,m+1}, \quad -n \leq m < n \\ FW_{n,-n} &= 0 & FW_{n,m} &= -\frac{b_{\lambda_{n,m}^+}(\kappa_-)}{b_{\lambda_{n,m-1}^+}(\kappa_-)} W_{n,m-1}, \quad -n < m \leq n. \end{aligned}$$

Theorem D. *The extended W -algebra \mathcal{M}_{p_+,p_-} is generated by the fields $T(z), Y(W_{1,1}; z)$ and $Y(W_{1,-1}; z)$. Furthermore*

1. *The soliton vector $W_{0,0}$ generates $\mathcal{X}_{1,1}^+$ as an \mathcal{M}_{p_+,p_-} -submodule of $\mathcal{K}_{1,1}^+$.*
2. *The module $\mathcal{X}_{1,1}^+$ is simple as an \mathcal{M}_{p_+,p_-} -module.*
3. *The quotient $\mathcal{K}_{1,1}^+/\mathcal{X}_{1,1}^+ \cong L(0)$ is an irreducible \mathcal{M}_{p_+,p_-} -module on which $Y(A; z) = 0$ for $A \in \mathcal{X}_{1,1}^+$.*
4. *The sequence*

$$0 \longrightarrow \mathcal{X}_{1,1}^+ \longrightarrow \mathcal{K}_{1,1}^+ \longrightarrow L(0) \longrightarrow 0,$$

is an exact sequence of \mathcal{M}_{p_+,p_-} -modules.

In order to prove the above Theorem we first state the following proposition.

Proposition 5.4. *1. For $n \geq 0$ and $k \geq -1$*

$$\begin{aligned} W_{1,-1}[\Delta_{n+k} - \Delta_n]W_{n+k,1-n} &= a_{n,k}W_{n,-n} \\ W_{1,1}[\Delta_{n+k} - \Delta_n]W_{n+k,n-1} &= b_{n,k}W_{n,n}, \end{aligned}$$

where

$$\begin{aligned} a_{n,k} &= \prod_{s \in \lambda_{n+k,1-n}^+} \frac{2(2p_+ - 1) + \kappa_- a'(s) - \ell'(s)}{\kappa_-(a(s) + 1) + \ell(s)} \\ b_{n,k} &= (-1)^{k+1} \frac{b_{\lambda_{1,1}^+}(\kappa_-)}{b_{\lambda_{n,n}^+}(\kappa_-)} \prod_{s \in \lambda_{n+k,n-1}^-} \frac{2(2p_- - 1) + \kappa_+ a'(s) - \ell'(s)}{\kappa_+ a(s) + \ell(s) + 1} \end{aligned}$$

2. For $m > \Delta_1 - \Delta_0$

$$W_{1,-1}[m]W_{1,1} = 0$$

$$W_{1,1}[m]W_{1,-1} = 0$$

Proof. We consider the case $W_{1,-1}[\Delta_{n+1} - \Delta_n]W_{n+1,1-n}$, the remaining cases follow by the same argument. In terms of screening operators $W_{1,-1}[\Delta_{n+1} - \Delta_n]W_{n+1,1-n}$ is given by the z -degree $\Delta_n - \Delta_{n+1} - \Delta_1$ term of

$$\int_{[\Gamma_{p_+-1}^+]} S_+(x_1) \cdots S_+(x_{p_+-1}) V_{\alpha_1^+}(z) dx \int_{[\Gamma_{3p_+-1}^+]} S_+(y_1) \cdots S_+(y_{3p_+-1}) |\alpha_{n+1}^+\rangle dy.$$

The z -degree $\Delta_n - \Delta_{n+1} - \Delta_1$ term of

$$V_{\alpha_1^+}(z) \int_{[\Gamma_{3p_+-1}^+]} S_+(y_1) \cdots S_+(y_{3p_+-1}) |\alpha_{n+1}^+\rangle dy$$

must be proportional to $|\alpha_n^+\rangle$ since that is the only vector with the correct conformal weight in $F_{\alpha_n^+}$. Note that this implies that there can be no terms of higher z -degree, which implies the second part of the Lemma for $n = 0$.

By the above reasoning the constant on the right hand side of

$$W_{1,-1}[\Delta_{n+1} - \Delta_n]W_{n+1,1-n} = \text{const} \cdot W_{n,-n}$$

is given by

$$\begin{aligned} & z^{\Delta_1 + \Delta_{n+1} - \Delta_n} \int_{[\Gamma_{3p_+-1}^+]} \langle \alpha_n^+ | V_{\alpha_1^+}(z) S_+(y_1) \cdots S_+(y_{3p_+-1}) | \alpha_{n+1}^+ \rangle \\ &= z^{\Delta_1 + \Delta_{n+1} - \Delta_n} \int_{[\Gamma_{3p_+-1}^+]} z^{2(n-1)p_-(2p_+-1)} \Delta_{3p_+-1}(y; \kappa_-) \times \\ & \quad \prod_{i=1}^{3p_+-1} y_i^{1-(2n+1)p_-} \prod_{i=1}^{3p_+-1} (1 - \frac{y_i}{z})^{2\kappa_+(1-2p_+)} dy \\ &= \varepsilon_{2(2p_+-1)} (Q_{\lambda_{n+1,1-n}^+}(w; \kappa_-)) = \prod_{s \in \lambda_{n+1,1-n}^+} \frac{2(2p_+-1) + \kappa_- a'(s) - \ell'(s)}{\kappa_-(a(s) + 1) + \ell(s)}. \end{aligned}$$

The remaining cases follow in the same way by computing

$$\begin{aligned} & z^{\Delta_1 + \Delta_{n+k} - \Delta_n} \int_{[\Gamma_{(2+k)p_+-1}^+]} \langle \alpha_n^+ | V_{\alpha_1^+}(z) S_+(y_1) \cdots S_+(y_{(2+k)p_+-1}) | \alpha_{n+k}^+ \rangle dy \\ &= \varepsilon_{2(2p_+-1)} (Q_{\lambda_{n+k,1-n}^+}(w; \kappa_-)) = \prod_{s \in \lambda_{n+k,1-n}^+} \frac{2(2p_+-1) + \kappa_- a'(s) - \ell'(s)}{\kappa_-(a(s) + 1) + \ell(s)} \end{aligned}$$

for $W_{1,-1}[\Delta_{n+k} - \Delta_n]W_{n+k,1-n}$, $k \geq -1$ and

$$\begin{aligned} & z^{\Delta_1 + \Delta_{n+k} - \Delta_n} \int_{[\Gamma_{(2+k)p_- - 1}]} \langle \alpha_n^- | V_{\alpha_1^-}(z) S_-(y_1) \cdots S_-(y_{(2+k)p_- - 1}) | \alpha_{n+k}^- \rangle dy \\ &= \varepsilon_{2(2p_- - 1)}(Q_{\lambda_{n+k,n-1}^-}(w; \kappa_+)) = \prod_{s \in \lambda_{n+k,n-1}^-} \frac{(2(2p_- - 1) + \kappa_+ a'(s) - \ell'(s))}{\kappa_+ a(s) + \ell(s) + 1} \end{aligned}$$

for $W_{1,1}[\Delta_{n+k} - \Delta_n]W_{n+k,n-1}$, $k \geq -1$. \square

Proof of Theorem D. Since $W_{0,0}$ is a Virasoro descendant of the vacuum $|0\rangle$, proving that \mathcal{M}_{p_+,p_-} is generated by $T(z)$, $Y(W_{1,-1}; z)$ and $Y(W_{1,1}; z)$, reduces to showing that all soliton vectors $W_{n,m}$ can be reached by acting on $W_{0,0}$ with the modes of $Y(W_{1,-1}; z)$ and $Y(W_{1,1}; z)$. Corollary 5.1 implies that $Y(W_{1,-1}; z)$ commutes with F and $Y(W_{1,1}; z)$ with E , it therefore follows by induction from Proposition 5.4 that $T(z)$, $Y(W_{1,-1}; z)$ and $Y(W_{1,1}; z)$ generate \mathcal{M}_{p_+,p_-} . It also follows that $\mathcal{X}_{1,1}^+$ is a simple \mathcal{M}_{p_+,p_-} submodule of $\mathcal{K}_{1,1}^+$, since any soliton vector of $\mathcal{X}_{1,1}^+$ can be reached from any other soliton vector by acting with the modes of $T(z)$, $Y(W_{1,-1}; z)$ and $Y(W_{1,1}; z)$.

Since $\mathcal{K}_{1,1}^+$ and $\mathcal{X}_{1,1}^+$ are \mathcal{M}_{p_+,p_-} -modules their quotient $\mathcal{K}_{1,1}^+/\mathcal{X}_{1,1}^+ \cong L(0)$ is an \mathcal{M}_{p_+,p_-} -module on which all fields $Y(A; z)$, $A \in \mathcal{X}_{1,1}^+$ act trivially. Thus the sequence

$$0 \longrightarrow \mathcal{X}_{1,1}^+ \longrightarrow \mathcal{K}_{1,1}^+ \longrightarrow L(0) \longrightarrow 0,$$

is an exact sequence of \mathcal{M}_{p_+,p_-} -modules. \square

Proposition 5.5. *Let $\omega_n(\beta)$ be the eigenvalues of the zero modes $W_{n,0}[0]$ acting on the generating state $|\beta\rangle \in F_\beta$ for $\beta \in \mathbb{C}$ and $n \geq 0$.*

1. *For $n \geq 0$ the eigenvalue $\omega_n(\beta)$ is a degree Δ_n polynomial in β*

$$\omega_n(\beta) = \langle \beta | W_{n,0}[0] | \beta \rangle = \prod_{i=1}^{(n+1)p_+ - 1} \prod_{j=1}^{(n+1)p_- - 1} \frac{\beta - \beta_{i,j}}{\beta_{(n+1)p_+ - i, 1 + j - (n+1)p_-}}.$$

2. *The polynomials $\omega_0(\beta)$, $\omega_1(\beta)^2$, $\omega_2(\beta)$ can be written as the following*

polynomials in $h_\beta = \frac{1}{2}\beta(\beta - \alpha_0)$

$$\begin{aligned}
g_0(h_\beta) &= \omega_0(\beta) = \text{const} \cdot \prod_{[(i,j)] \in \mathcal{T}} (h_\beta - \Delta_{i,j}), \\
g_1(h_\beta) &= \omega_1(\beta)^2 = \text{const} \cdot \prod_{[(i,j)] \in \mathcal{T}} (h_\beta - \Delta_{i,j})^4 \prod_{i=1}^{p_+-1} \prod_{j=1}^{p_--1} (h_\beta - \Delta_{i,j;0}^+)^2 \\
&\quad \prod_{i=1}^{p_+-1} (h_\beta - \Delta_{i,p_-;0}^+)^2 \prod_{j=1}^{p_--1} (h_\beta - \Delta_{p_+,j;0}^+)^2 \\
&\quad (h_\beta - \Delta_{p_+,p_-;0}^+), \\
g_2(h_\beta) &= \omega_2(\beta) = \text{const} \cdot \prod_{[(i,j)] \in \mathcal{T}} (h_\beta - \Delta_{i,j})^3 \\
&\quad \prod_{i=1}^{p_+-1} \prod_{j=1}^{p_--1} (h_\beta - \Delta_{i,j;0}^+)^2 \prod_{i=1}^{p_+-1} \prod_{j=1}^{p_--1} (h_\beta - \Delta_{i,j;0}^-)^2 \\
&\quad \prod_{i=1}^{p_+-1} (h_\beta - \Delta_{i,p_-;0}^+)^2 \prod_{i=1}^{p_+-1} (h_\beta - \Delta_{i,p_-;0}^-)^2 \\
&\quad \prod_{j=1}^{p_--1} (h_\beta - \Delta_{p_+,j;0}^+)^2 \prod_{j=1}^{p_--1} (h_\beta - \Delta_{p_+,j;0}^-)^2 \\
&\quad (h_\beta - \Delta_{p_+,p_-;0}^+)(h_\beta - \Delta_{p_+,p_-;0}^-),
\end{aligned}$$

Where \mathcal{T} is the Kac table and $\Delta_{r,s}$ and $\Delta_{r,s;0}^\pm$ are the conformal weights of Definition 4.10.

Proof. The first statement can be expressed in terms of Jack polynomials and evaluated accordingly.

$$\begin{aligned}
&\int_{\Gamma_{(n+1)p_+-1}^+} z^{\Delta_n} \langle \beta | S_+(y_1 + z) \cdots S_+(y_{(n+1)p_+-1} + z) V_{\alpha_n^+}(z) | \beta \rangle dy \\
&= z^{\Delta_n} \int_{\Gamma_{(n+1)p_+-1}^+} \Delta_{(n+1)p_+-1}(y; \kappa_-) \prod_{i=1}^{(n+1)p_+-1} y_i^{1-(n+1)p_-} \prod_{i=1}^{(n+1)p_+-1} (1 + \frac{y_i}{z})^{\alpha+\beta} \frac{dy}{y} \\
&= \varepsilon_{\alpha-\beta}(Q_{\lambda_{n,0}^+}(w; \kappa_-)) = \prod_{i=1}^{(n+1)p_+-1} \prod_{j=1}^{(n+1)p_--1} \frac{\beta - \beta_{i,j}}{\beta_{(n+1)p_+-i,1+j-(n+1)p_-}}
\end{aligned}$$

The second statement follows from the first by using the identity

$$(\beta - \beta_{i,j})(\beta - \beta_{-i,-j}) = \beta^2 - \beta(\beta_{i,j} + \beta_{-i,-j}) + \beta_{i,j}\beta_{-i,-j} = 2(h_\beta - h_{i,j}).$$

□

5.3 The zero mode algebra and the c_2 -cofiniteness condition

In this section we will determine the structure of the zero mode algebra $A_0(\mathcal{M}_{p_+,p_-})$ and the Poisson algebra $\mathfrak{p}(\mathcal{M}_{p_+,p_-})$, as well as prove that \mathcal{M}_{p_+,p_-} satisfies Zhu's c_2 -cofiniteness condition. Let $A_0 = A_0(\mathcal{M}_{p_+,p_-})$. We will first compute relations for the zero mode algebra A_0 and then show that they imply that the Poisson algebra is finite dimensional and that therefore \mathcal{M}_{p_+,p_-} satisfies Zhu's c_2 -cofiniteness condition.

Proposition 5.6. *1. The \mathcal{M}_{p_+,p_-} derivations E, F, H are also derivations of A_0 .*

2. The \mathcal{M}_{p_+,p_-} anti-involution σ defines an anti-involution of A_0 satisfying

$$\sigma([T]) = [T] \quad \sigma([W_{1,m}]) = -[W_{1,m}].$$

3. We have the following surjection onto the zero mode algebra A_0 from a subspace of $\mathcal{K}_{1,1}^+$

$$\mathbb{C}[T] \oplus \bigoplus_{m=-1}^1 \mathbb{C}[T] * [W_{1,m}] \rightarrow A_0,$$

where $\mathbb{C}[T]$ denotes polynomials in $[T]$ with multiplication $*$.

Proof. The first two statements follow because they hold in $U[0]$ and descend to A_0 .

We prove the third statement by using Zhu's formulation of the zero mode algebra. It follows from Lemma 5.4 and Proposition 2.4 that for $n \geq 2$

$$W_{n,n} = \text{const} \cdot W_{1,1}[\Delta_{n-1} - \Delta_n]W_{n-1,n-1} \in O(\mathcal{M}_{p_+,p_-}).$$

By applying F multiple times to $W_{n,n}$ we see that $W_{n,m} \in O(\mathcal{M}_{p_+,p_-})$ for $n \geq 2$ and $-n \leq m \leq n$. From Proposition 2.4 it also follows that the image of L_{-n} for $n \geq 3$ satisfies

$$L_{-n}A \in \mathbb{C}[L_{-2}, L_{-1}, L_0]A \bmod O(\mathcal{M}_{p_+,p_-}).$$

Since $(L_0 + L_{-1})A \in O(\mathcal{M}_{p_+,p_-})$ we therefore have the following surjection of vector spaces

$$\mathbb{C}[L_{-2}]\lvert 0 \rangle \oplus \bigoplus_{m=-1}^1 \mathbb{C}[L_{-2}]W_{1,m} \longrightarrow A_0.$$

Therefore A_0 is spanned by polynomials in T and polynomials in T times $W_{1,m}$. \square

Theorem E. *The zero mode algebra A_0 is finite dimensional and satisfies:*

1. *The zero mode algebra A_0 is generated by $[T], [W_{1,m}]$, $m = -1, 0, 1$.*
2. *The Virasoro element satisfies the polynomials relations*

$$\begin{aligned} [W_{0,0}] &= g_0([T]) & [W_{1,0}]^2 &= g_1([T]) \\ [W_{2,0}] &= g_2([T]) = 0, \end{aligned}$$

where the g_i are the polynomials of Proposition 5.5.

3. *The products of the generators $[W_{1,m}]$, $m = -1, 0, 1$ are given by*

	$[W_{1,-1}]$	$[W_{1,0}]$	$[W_{1,1}]$
$[W_{1,-1}]$	0	$-f([T])[W_{1,-1}]$	$-g_1([T]) - f([T])[W_{1,0}]$
$[W_{1,0}]$	$f([T])[W_{1,-1}]$	$g_1([T])$	$-f([T])[W_{1,1}]$
$[W_{1,1}]$	$-g_1([T]) + f([T])[W_{1,0}]$	$f([T])[W_{1,1}]$	0

where f is a degree less than $\Delta_1/2$ polynomial and the g_i are the polynomials of Proposition 5.5.

4. *The commutators of generators $[W_{1,m}]$, $m = -1, 0, 1$ are given by*

$$\begin{aligned} [[W_{1,0}], [W_{1,1}]] &= -2f([T])[W_{1,1}], \quad [[W_{1,0}], [W_{1,-1}]] = 2f([T])[W_{1,-1}], \\ [[W_{1,1}], [W_{1,-1}]] &= 2f([T])[W_{1,0}], \end{aligned}$$

Proof. Since $[W_{2,0}]$ vanishes in A_0 , we obtain $g_2([T]) = 0$ from Proposition 5.5. The finite dimensionality of A_0 then follows from Proposition 5.6 and $g_2([T]) = 0$.

The relation $[W_{1,\pm 1}]^2 = 0$ follows from the fact that $W_{1,\pm 1} * W_{1,\pm 1} = 0$ in $\mathcal{K}_{1,1}^+$.

To show that $[W_{1,0}]^2 = g_1([T])$ we recall that

$$W_{1,0} * W_{1,0} \in U(\mathcal{L})|0\rangle \oplus U(\mathcal{L})W_{1,0} \subset \mathcal{K}_{1,1}^+.$$

This implies that in A_0

$$[W_{1,0}] \cdot [W_{1,0}] = g([T]) + \tilde{g}([T])[W_{1,0}].$$

By applying σ to both sides we get

$$[W_{1,0}] \cdot [W_{1,0}] = g([T]) - \tilde{g}([T])[W_{1,0}],$$

therefore $\tilde{g} = 0$ and $g = g_1$ by Proposition 5.5.

To show the relation for $[W_{1,0}] \cdot [W_{1,-1}]$ recall that

$$W_{1,0} * W_{1,-1} \in U(\mathcal{L})W_{1,-1} \subset \mathcal{K}_{1,1}^+.$$

This implies that in A_0

$$[W_{1,0}] \cdot [W_{1,-1}] = f([T])[W_{1,-1}].$$

for some degree less than $\Delta_1/2$ polynomial f . We take this to be the polynomial f in the theorem. The remaining relations follow by the application of E and σ to the above relations. \square

Theorem F. 1. *The Poisson algebra $\mathfrak{p}(\mathcal{M}_{p_+,p_-})$ is finite dimensional and \mathcal{M}_{p_+,p_-} therefore satisfies Zhu's c_2 -cofiniteness condition.*

2. *As a commutative algebra $\mathfrak{p}(\mathcal{M}_{p_+,p_-})$ is generated by $[T], [W_{1,-1}], [W_{1,0}]$ and $[W_{1,1}]$.*

3. *The maps E, F and H act as derivations on $\mathfrak{p}(\mathcal{M}_{p_+,p_-})$.*

4.

$$\begin{aligned} [W_{1,\pm 1}]_{\mathfrak{p}}^2 &= 0 \\ [W_{1,1}]_{\mathfrak{p}} \cdot [W_{1,-1}]_{\mathfrak{p}} &= -[W_{1,0}]_{\mathfrak{p}}^2 \\ [W_{1,0}]_{\mathfrak{p}}^2 &= \text{const} \cdot [T]_{\mathfrak{p}}^{\Delta_1} \end{aligned}$$

Proof. The second statement follows from Theorem D and the third from Corollary 5.1.

The first relation of the fourth statement follows from

$$W_{1,\pm 1}[-\Delta_1]W_{1,\pm 1} = 0.$$

the second relation follows by applying E^2 to $[W_{1,-1}]^2 = 0$. The third relation is a consequence of Proposition 2.11 The relation $[W_{1,0}]^2 = g_1([T])$ in A_0 becomes

$$[W_{1,0}] = \text{const} \cdot [T]^{\Delta_1}$$

in $\text{Gr}_{\Delta_1}(A_0)$ and therefore the same relation follows in $\mathfrak{p}(\mathcal{M}_{p_+,p_-})$.

The Poisson algebra $\mathfrak{p}(\mathcal{M}_{p_+,p_-})$ is finite dimensional because it is finitely generated and that all of its generators are nilpotent. \square

5.4 Classification of simple \mathcal{M}_{p_+, p_-} modules

In this section we will classify all simple modules of the zero mode algebra and of \mathcal{M}_{p_+, p_-} . We will see that all simple \mathcal{M}_{p_+, p_-} -modules can be either realised by minimal model modules $L_{r,s}$ or submodules of the lattice modules $V_{r,s}^\pm$.

Proposition 5.7. *Let $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ be the Virasoro modules of Proposition 4.12, where $1 \leq r \leq p_+$, $1 \leq s \leq p_-$.*

1. *The $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ are \mathcal{M}_{p_+, p_-} -modules.*
2. *The $\mathcal{X}_{r,s}^\pm$ are simple \mathcal{M}_{p_+, p_-} -modules for $1 \leq r \leq p_+$, $1 \leq s \leq p_-$.*
3. *For $1 \leq r < p_+$, $1 \leq s < p_-$ the modules $\mathcal{K}_{r,s}^+$ satisfy the exact sequences*

$$0 \rightarrow \mathcal{X}_{r,s}^+ \rightarrow \mathcal{K}_{r,s}^+ \rightarrow L_{[r,s]} \rightarrow 0,$$

where the arrows are \mathcal{M}_{p_+, p_-} -homomorphisms and

$$L_{[r,s]} = L_{[p_+ - r, p_- - s]} = \mathcal{K}_{r,s}^+ / \mathcal{X}_{r,s}^+ = L(h_{r,s}; 0)$$

are the simple modules of the minimal model VOA $\text{MinVir}_{c_{p_+, p_-}}$.

4. *For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$ the spaces of lowest conformal weight $\mathcal{K}_{r,s}^+[h_{r,s}; 0]$ and $\mathcal{X}_{r,s}^+[\Delta_{r,s}^+]$ of $\mathcal{K}_{r,s}^+$ and $\mathcal{X}_{r,s}^+$ are one dimensional and the zero modes $W_{1,m}[0]$, $m = -1, 0, 1$ act trivially. The spaces of lowest conformal weight $\mathcal{K}_{r,s}^-[\Delta_{r,s}^-] = \mathcal{X}_{r,s}^-[\Delta_{r,s}^-]$ of $\mathcal{K}_{r,s}^- = \mathcal{X}_{r,s}^-$ are two dimensional and the zero modes $W_{1,m}[0]$, $m = -1, 0, 1$ act non-trivially.*

Proof. The fact that $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ are \mathcal{M}_{p_+, p_-} -modules follows analogously to the proof of Theorem D by showing that $Y(W_{1,\pm 1}; z)$ act transitively on the soliton vectors. This then also implies that the $\mathcal{X}_{r,s}^\pm$ are simple.

The $L_{[r,s]}$ are \mathcal{M}_{p_+, p_-} -modules, because they are quotients of \mathcal{M}_{p_+, p_-} -modules. They are simple, because they are already simple as Virasoro modules.

The spaces of lowest conformal weight of $\mathcal{K}_{r,s}^+$ are just $\mathbb{C}|\beta_{r,s}; 0\rangle$ while for $\mathcal{X}_{r,s}^\pm$ they are $V_{r,s;0}^\pm$. \square

For a simple \mathcal{M}_{p_+, p_-} -module M let \overline{M} be its corresponding simple A_0 -module.

Proposition 5.8. *The simple modules of the zero mode algebra corresponding to the simple \mathcal{M}_{p_+, p_-} -modules of Proposition 5.7 have the following structure:*

1. For $1 \leq r < p_+$, $1 \leq s < p_-$ the simple A_0 -modules $\overline{L_{[r,s]}} = \overline{L_{[p_+-r, p_-s]}}$ are 1-dimensional. The Virasoro $[T]$ element acts as $\Delta_{r,s} \cdot \text{id}$, while the $[W_{1,m}]$ act trivially.
2. For $1 \leq r < p_+$, $1 \leq s < p_-$, the simple A_0 -modules $\overline{\mathcal{X}_{r,s}^+}$ are 1-dimensional. The $[W_{1,m}]$ act trivially, while the Virasoro element acts as

$$[T] = \Delta_{r,s;0}^+ \cdot \text{id}.$$

3. For $1 \leq r < p_+$, $1 \leq s < p_-$, the simple A_0 -modules $\overline{X_{r,s}^-}$ are 2-dimensional. Let v_{\pm} be a basis such that

$$v_- = [W_{1,-1}]v_+,$$

then

$$\begin{aligned} [T]v_{\pm} &= \Delta_{r,s;0}^- v_{\pm} \\ [W_{1,0}]v_{\pm} &= \mp f([T])v_{\pm} \end{aligned}$$

and $\Delta_{r,s;0}^-$ is not a root of $f(h_{\beta})$.

Proof. Recall the structure of the $\mathcal{K}_{r,s}^{\pm}$ and $\mathcal{X}_{r,s}^{\pm}$ as laid out in Proposition 4.12.

1. For $1 \leq r < p_+$, $1 \leq s < p_-$ the factorisation of $\omega_1(\beta)$ in Proposition 5.5 contains a factor $h_{\beta} - \Delta_{r,s}$. Therefore the A_0 generator $[W_{1,0}]$ acts trivially on $\overline{L_{[r,s]}}$. By applying E and F one sees that $[W_{1,\pm 1}]$ must also act trivially. Thus $[T]$ is the only generator of A_0 that acts non-trivially on $\overline{L_{[r,s]}}$ and it acts by multiplying by the conformal weight $\Delta_{r,s}$.
2. The case for $\overline{\mathcal{X}_{r,s}^+}$ follows in the same way as for $\overline{L_{[r,s]}}$.
3. For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, the zero modes $W_{1,m}[0]$, $m = -1, 0, 1$ act non-trivially on the two dimensional soliton sector $V_{r,s;0}^- \subset \mathcal{X}_{r,s}^-$. Therefore there exists a basis v_{\pm} of $\overline{\mathcal{X}_{r,s}^-}$ such that

$$v_- = [W_{1,-1}]v_+ \quad v_+ = \text{const}[W_{1,1}]v_-.$$

It then follows from the relations of A_0 and the conformal weight of $V_{r,s;0}^-$ that

$$\begin{aligned} [T]v_{\pm} &= \Delta_{r,s;0}^- v_{\pm} \\ [W_{1,0}]v_{\pm} &= \mp f([T])v_{\pm}. \end{aligned}$$

Since $[W_{1,0}]$ acts non-trivially on $\overline{X_{r,s}^-}$, the conformal weight $\Delta_{r,s}^-$ cannot be a root of $f(h_{\beta})$.

□

Theorem G. *The list of $2p_+p_- + (p_+ - 1)(p_- - 1)/2$ simple \mathcal{M}_{p_+,p_-} -modules and simple A_0 -modules of Propositions 5.7 and 5.8 are a full classification of all simple \mathcal{M}_{p_+,p_-} - and A_0 -modules.*

Proof. From Theorem E we know that $g_2([T]) = 0$ in A_0 . Therefore the minimal polynomial of $[T]$ must divide $g_2([T])$, that is the conformal weight of any simple A_0 -module must be a zero of $g_2(h_\beta)$. The list of simple A_0 -modules in Proposition 5.8 exhausts all these possibilities, therefore there can be no other inequivalent modules other than those listed. □

5.5 The Whittaker category \mathcal{M}_{p_+,p_-} -Whitt-mod

Let $\mathcal{M}_{p_+,p_-}\text{-mod}$ be the category of left \mathcal{M}_{p_+,p_-} -modules. Then in the sense of abelian categories any object M of $\mathcal{M}_{p_+,p_-}\text{-mod}$ has a finite Jordan-Hölder composition series and the simple objects of $\mathcal{M}_{p_+,p_-}\text{-mod}$ are given by

$$\{L_{[r,s]} | 1 \leq r < p_+, 1 \leq s < p_-\} \cup \{\mathcal{X}_{r,s}^\pm | 1 \leq r \leq p_+, 1 \leq s \leq p_-\}.$$

Definition 5.9. *We define the full subcategory \mathcal{M}_{p_+,p_-} -Whitt-mod of $\mathcal{M}_{p_+,p_-}\text{-mod}$ as the category of all objects $M \in \mathcal{M}_{p_+,p_-}\text{-mod}$ such that the composition series of M only contains simple objects in*

$$\{\mathcal{X}_{r,s}^\pm | 1 \leq r \leq p_+, 1 \leq s \leq p_-\}.$$

We call the category \mathcal{M}_{p_+,p_-} -Whitt-mod the Whittaker category of $\mathcal{M}_{p_+,p_-}\text{-mod}$ and the objects of \mathcal{M}_{p_+,p_-} -Whitt-mod Whittaker \mathcal{M}_{p_+,p_-} -modules.

The full subcategory \mathcal{M}_{p_+,p_-} -Whitt-mod has properties crucial to developing the conformal field theory associated to \mathcal{M}_{p_+,p_-} .

6 Concluding remarks and further problems

Since the VOA \mathcal{M}_{p_+,p_-} satisfies the c_2 -cofiniteness condition, one can develop the conformal field theory over general Riemann surfaces associated to \mathcal{M}_{p_+,p_-} . A paper on this subject is being prepared by the first author together with Hashimoto [HT].

We have some conjectures regarding the conformal field theory over general Riemann surfaces associated to \mathcal{M}_{p_+,p_-} . By considering the conformal field theory on the Riemann sphere associated to \mathcal{M}_{p_+,p_-} , the fusion tensor product \otimes induces the structure of a braided monoidal category on $\mathcal{M}_{p_+,p_-}\text{-mod}$. However, as was noted in [GRW], this fusion tensor product is not exact on $\mathcal{M}_{p_+,p_-}\text{-mod}$. We conjecture the following:

1. The full abelian subcategory $\text{Vir}_{\min} \subset \mathcal{M}_{p_+, p_-}\text{-mod}$, generated by the simple modules

$$\{L_{[r,s]} = |1 \leq r < p_+, 1 \leq s < p_-\}$$

forms a tensor ideal in $\mathcal{M}_{p_+, p_-}\text{-mod}$. Thus $\mathcal{M}_{p_+, p_-}\text{-Whitt-mod}$ is endowed with a quotient braided monoidal structure $(\mathcal{M}_{p_+, p_-}\text{-Whitt-mod}, \dot{\otimes})$.

2. The category $(\mathcal{M}_{p_+, p_-}\text{-Whitt-mod}, \dot{\otimes})$ is rigid as a monoidal category.
3. In their seminal paper [FGSTa] Feigin, Gainutdinov, Semikhatov and Tipunin defined the quantum group \mathfrak{g}_{p_+, p_-} – a finite dimensional complex Hopf algebra. They determined all simple modules of \mathfrak{g}_{p_+, p_-} and their projective covers. There are exactly $2p_+p_-$ simple modules, which is also the number of simple objects in $\mathcal{M}_{p_+, p_-}\text{-Whitt-mod}$. The Hopf algebra \mathfrak{g}_{p_+, p_-} is not quasi-triangular, nevertheless, we expect the monoidal category $(\mathfrak{g}_{p_+, p_-}\text{-mod}, \dot{\otimes})$ to be braided. Since \mathfrak{g}_{p_+, p_-} is a Hopf algebra, $(\mathfrak{g}_{p_+, p_-}\text{-mod}, \dot{\otimes})$ is a rigid monoidal category. We therefore conjecture that

$$(\mathcal{M}_{p_+, p_-}\text{-Whitt-mod}, \dot{\otimes}) \cong (\mathfrak{g}_{p_+, p_-}\text{-mod}, \dot{\otimes})$$

as braided monoidal categories.¹

4. In [FGSTa] it was also shown that the centre $Z(\mathfrak{g}_{p_+, p_-})$ of \mathfrak{g}_{p_+, p_-} is $\frac{1}{2}(3p_+ - 1)(3p_- - 1)$ dimensional and carries the structure of a $\text{SL}(2, \mathbb{Z})$ -module. The space of vacuum amplitudes of $\mathcal{M}_{p_+, p_-}\text{-mod}$ on the torus also carries an $\text{SL}(2, \mathbb{Z})$ action. Explaining the relation between these two $\text{SL}(2, \mathbb{Z})$ modules will be an important future problem.
5. The universal enveloping algebra $U(\mathcal{M}_{p_+, p_-})$ contains the elements $L_0, W_{n,0}[0], n \geq 0$. We conjecture that these elements are mutually commuting in $U(\mathcal{M}_{p_+, p_-})$.
6. The Frobenius currents $E(z), F(z)$ and $H(z)$ in $\text{End}(\mathcal{M}_{p_+, p_-})[[z, z^{-1}]]$ are local to each other. We conjecture that they satisfy the relations of an affine Lie algebra of type \mathfrak{sl}_2 at level 0. So by the adjoint action the universal enveloping algebra $U(\mathcal{M}_{p_+, p_-})$ carries the structure of a representation of an affine Lie algebra of type \mathfrak{sl}_2 at level 0. We conjecture that this representation has the crystal structure of [Kas].

¹The first author attributes this conjecture to exciting discussions with Semikhatov and Tipunin during his stay in Moscow in January 2013 and is convinced of its validity.

7. For any simple Lie algebra \mathfrak{g} of ADE type and any pair of coprime integers $p_+, p_- \geq h$, where h is the Coxeter number of \mathfrak{g} , one can define an extended W -algebra $\mathcal{M}_{p_+, p_-}(\mathfrak{g})$ of type \mathfrak{g} at positive fractional level. If $\mathfrak{g} = \mathfrak{sl}_2$ then $\mathcal{M}_{p_+, p_-}(\mathfrak{sl}_2) = \mathcal{M}_{p_+, p_-}$. We conjecture that these extended algebras will have a similar structure to \mathcal{M}_{p_+, p_-} such as c_2 -cofiniteness; 2ℓ screening operators, where ℓ is the rank of \mathfrak{g} ; Frobenius currents satisfying the relations of affine \mathfrak{g} at level 0; and so on.

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